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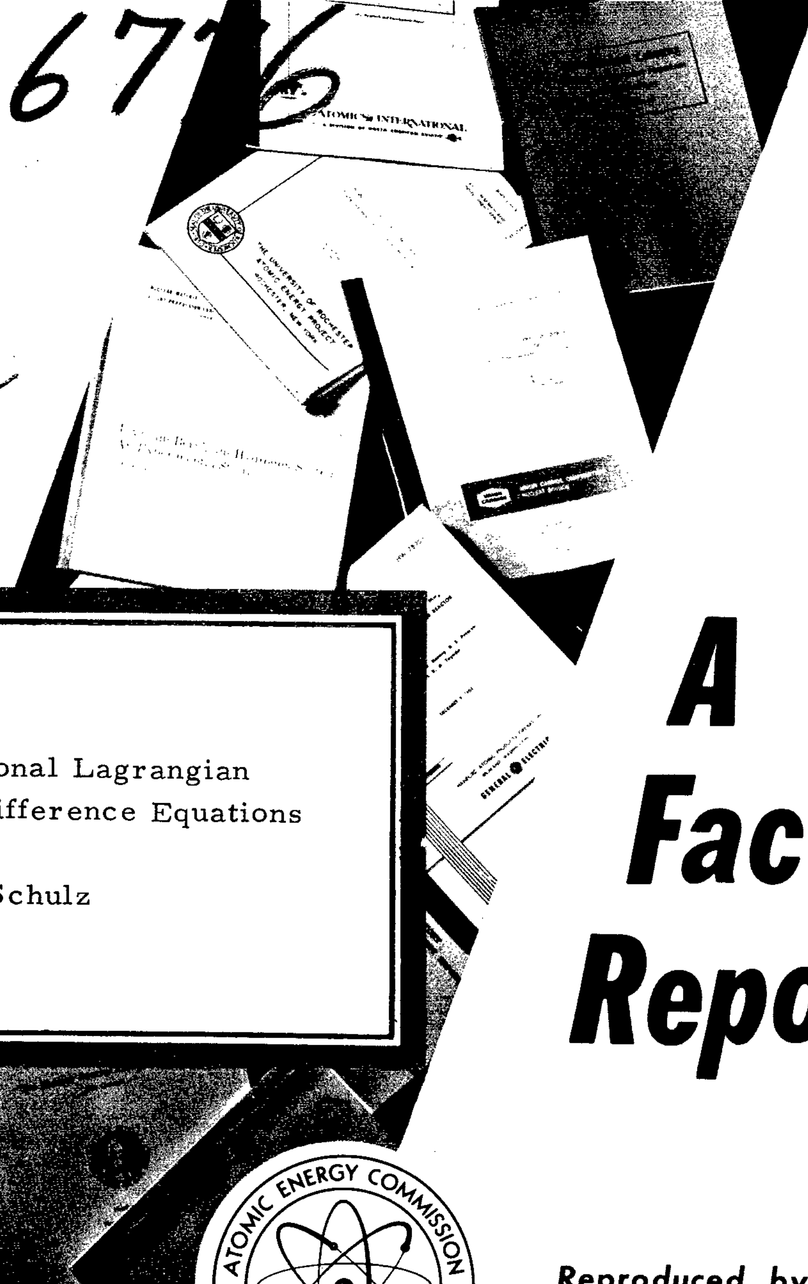

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Two-Dimensional Lagrangian Hydrodynamic Difference Equations

William D. Schulz

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TWO-DIMENSIONAL LAGRANGIAN HYDRODYNAMIC DIFFERENCE EQUATIONS

William D. Schulz

April 17, 1963

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CONTENTS

Two-Dimensional Lagrangian Hydrodynamic Difference Equations

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Livermore, California

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ABSTRACT

A numerical model of a fluid is constructed by adding a tensor artificial viscosity to the hydrodynamic differential equations which are then differenced to produce a set of equations suitable for solving on a digital computer. The equations apply to a system of two space coordinates and one time coordinate. The space coordinates can be either Cartesian or cylindrical.

Two-Dimensional Lagrangian Hydrodynamic Difference Equations*

William D. Schulz

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April 17, 1963

I. INTRODUCTION

For many years people at Livermore have been working on two-dimensional hydrodynamic problems (two space and one time coordinate). Early analytic attempts to solve these problems led nowhere and thus essentially the entire effort consists of converting the differential equations to difference equations and then solving the difference equations on a digital computer.

Some of the people who have been engaged in this business are: F. Bjorklund, R. Grandey, N. Hardy, R. Herbst, C. Leith, R. Lelevier, W. Noh, S. Sack, J. Trulio, and others.[1] R. Lelevier wrote the first two-dimensional hydro code eight or so years ago. The first production code was produced by S. Sack in collaboration with R. Lelevier. This paper will present some analysis and then a set of difference equations as used in a code by the author.

II. DEFINITIONS, NOTATION, AND TRANSFORMATION RELATIONS

For convenience a few relations connecting Eulerian coordinates with Lagrangian coordinates will be discussed first. Eulerian coordinates are the ones which are most familiar in physics. In addition, Eulerian expressions serve as shorthand for longer Lagrangian equivalents. Before differencing, of course, everything is put in Lagrangian form. (See Fig. 1.)

$R(k, \ell, q)$ = Eulerian coordinate, may be either Cartesian or cylindrical,

$Z(k, \ell, q)$ = Eulerian coordinate, always Cartesian,

$\underset{\sim}{R}$ = the vector (R,Z),

k = Lagrangian coordinate,

ℓ = Lagrangian coordinate,

j = Jacobian of transformation.

If $R_k = \frac{\partial R}{\partial k}$, $R_\ell = \frac{\partial R}{\partial \ell}$, etc., then $j = R_k Z_\ell - R_\ell Z_k$ = area Jacobian.

Let

$$\hat{R} = \begin{cases} R \text{ for cylindrical coordinates,} \\ 1 \text{ for Cartesian coordinates,} \end{cases}$$

then a "volume Jacobian" may be defined as

$$J = \hat{R} j.$$

Consider the conversion of an Eulerian derivative into a Lagrangian derivative:

$$\frac{\partial}{\partial R} = \left(\frac{\partial k}{\partial R}\right) \frac{\partial}{\partial k} + \left(\frac{\partial \ell}{\partial R}\right) \frac{\partial}{\partial \ell},$$

$$\frac{\partial}{\partial Z} = \left(\frac{\partial k}{\partial Z}\right) \frac{\partial}{\partial k} + \left(\frac{\partial \ell}{\partial Z}\right) \frac{\partial}{\partial \ell}.$$

Expressions for $\frac{\partial k}{\partial R}, \ldots, \frac{\partial \ell}{\partial Z}$, in terms of $R_k, \ldots, Z_\ell$ can be found as follows:

For arbitrary g,

$$\frac{\partial g}{\partial k} = R_k \frac{\partial g}{\partial R} + Z_k \frac{\partial g}{\partial Z},$$

$$\frac{\partial g}{\partial \ell} = R_\ell \frac{\partial g}{\partial R} + Z_\ell \frac{\partial g}{\partial Z}.$$

Let $g = k$, then one can solve for $\frac{\partial k}{\partial R}$ and $\frac{\partial k}{\partial Z}$. Similarly $g = \ell$ gives $\frac{\partial \ell}{\partial R}$ and $\frac{\partial \ell}{\partial Z}$. The result of this is

$$\frac{\partial k}{\partial R} = \frac{Z_\ell}{J}, \qquad \frac{\partial \ell}{\partial R} = -\frac{Z_k}{J},$$

$$\frac{\partial k}{\partial Z} = -\frac{R_\ell}{J}, \qquad \frac{\partial \ell}{\partial Z} = \frac{R_k}{J},$$

which gives

$$\frac{\partial}{\partial R} = \frac{Z_\ell}{J} \frac{\partial}{\partial k} - \frac{Z_k}{J} \frac{\partial}{\partial \ell},$$

$$\frac{\partial}{\partial Z} = -\frac{R_\ell}{J} \frac{\partial}{\partial k} + \frac{R_k}{J} \frac{\partial}{\partial \ell}.$$

Define a vector $\underset{\sim}{\bar{R}}$ lagging $\underset{\sim}{R}$ by 90° as the "normal vector" to $\underset{\sim}{R}$:

$$\underset{\sim}{\bar{R}} = (Z, -R) = \text{normal vector to } \underset{\sim}{R}.$$

Note that

$$\underset{\sim}{\bar{R}}_1 \cdot \underset{\sim}{\bar{R}}_2 = \underset{\sim}{R}_1 \cdot \underset{\sim}{R}_2, \quad \underset{\sim}{\bar{R}}_1 \cdot \underset{\sim}{R}_2 = -\underset{\sim}{R}_1 \cdot \underset{\sim}{\bar{R}}_2.$$

A useful vector operator can be defined as

$$\underset{\sim}{D} = \frac{1}{J}\left[\underset{\sim}{\bar{R}}_\ell \frac{\partial}{\partial k} - \underset{\sim}{\bar{R}}_k \frac{\partial}{\partial \ell}\right] = \frac{1}{J}\left[\frac{\partial}{\partial k}\left(\underset{\sim}{\bar{R}}_\ell \ldots\right) - \frac{\partial}{\partial \ell}\left(\underset{\sim}{\bar{R}}_k \ldots\right)\right].$$

Then

$$\nabla f = \underset{\sim}{D} f,$$

$$\nabla \cdot \underset{\sim}{f} = \frac{1}{\hat{R}} \underset{\sim}{D} \cdot (\hat{R} f).$$

Lagrangian time derivatives, that is, partial derivatives with k and ℓ held fixed, will be written as follows:

$$u(k, \ell, t) = \frac{\partial R}{\partial t} = R_t = \mathring{R} = R \text{ velocity},$$

$$v(k, \ell, t) = \frac{\partial Z}{\partial t} = Z_t = \mathring{Z} = Z \text{ velocity},$$

$$\underset{\sim}{u} = \text{the vector } (u, v).$$

At this point we can deduce some relations for later use:

$$j = \underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell ,$$

$$j_t = \underset{\sim}{R}_{kt} \cdot \overline{\underset{\sim}{R}}_\ell + \underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_{\ell t}$$

$$= \overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{R}_{kt} - \overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{R}_{\ell t} ,$$

$$j_t = \overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k - \overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell = j\underset{\sim}{D} \cdot \underset{\sim}{u} . \qquad (2.1)$$

In cylindrical coordinates

$$\nabla \cdot \underset{\sim}{u} = \frac{u}{R} + \underset{\sim}{D} \cdot \underset{\sim}{u} .$$

Substituting from (2.1) for $\underset{\sim}{D} \cdot \underset{\sim}{u}$,

$$\nabla \cdot \underset{\sim}{u} = \frac{R_t}{R} + \frac{j_t}{j} = \frac{(Rj)_t}{Rj} .$$

Therefore

$$\nabla \cdot \underset{\sim}{u} = \frac{J_t}{J} . \qquad (2.2)$$

Since J = j for Cartesian systems, (2.2) reduces to (2.1) and thus is valid for both coordinate systems.

To complete the discussion of the transformation relations, the connection between Eulerian and Lagrangian time derivatives may be deduced as follows:

Let g be an arbitrary function, then

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t}\Big|_{\underset{\sim}{R}} + \frac{\partial g}{\partial R} R_t + \frac{\partial g}{\partial Z} Z_t .$$

Thus

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\Big|_{\underset{\sim}{R}} + (\underset{\sim}{u} \cdot \nabla) .$$

However, except for the above equations, only Lagrangian time derivatives will appear in this report.

Some further notation and definitions:

p = pressure,

ρ = density,

$\tau = \rho^{-1}$ = specific volume,

ϵ = specific internal energy,

S = entropy,

$M = \rho^o J^o$ = mass constant (a superscript o means t = 0),

$\epsilon(s)$ = specific internal energy introduced through an external source,

$E = M\epsilon$.

III. CONSERVATION EQUATIONS

Mass Equation

$$\tau = \frac{J}{M} . \qquad (3.1)$$

This follows directly from the mass conservation equation as it is more usually written,

$$(\rho J)_t = 0,$$

and therefore

$$\rho J = \rho^o J^o = M.$$

Differential Momentum Equation

$$\rho \underset{\sim}{u}_t + \nabla p + \frac{\rho}{M}\left[\frac{\partial}{\partial k}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell q_A\right) - \frac{\partial}{\partial \ell}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k q_B\right)\right] = 0. \qquad (3.2)$$

The hydrodynamic equations use a tensor artificial viscosity to smooth out shock discontinuities. q_A is to be thought of as a "one-dimensional" viscosity associated with the $\overline{\underset{\sim}{R}}_\ell$ direction while q_B is a "one-dimensional" viscosity associated with the $\overline{\underset{\sim}{R}}_k$ direction.[2]

Internal Energy Equation

$$\frac{\partial \epsilon}{\partial t} - \frac{\partial \epsilon(s)}{\partial t} + p\frac{\partial \tau}{\partial t} + \frac{q_A\left(\bar{R}_\ell \cdot u_k\right)}{\rho J} + \frac{q_B\left(-\bar{R}_k \cdot u_\ell\right)}{\rho J} = 0. \quad (3.3)$$

The equation of total energy conservation (3.5) demonstrates the self-consistency of the terms containing q_A and q_B in the momentum and internal energy equations.

Equation of State

The quantity p which appears in the conservation equations is computed from a known function of ϵ and τ. Thus one has given equations (or tables) of the form

$$p = p(\epsilon, \tau). \quad (3.4)$$

Total Energy Conservation Equation

From the momentum and internal energy equations, we have

$$\rho u \cdot \left\{ u_t + \frac{\nabla p}{\rho} + \frac{1}{M}\left[\frac{\partial}{\partial k}\left(\hat{R}\bar{R}_\ell\, q_A\right) - \frac{\partial}{\partial \ell}\left(\hat{R}\bar{R}_k q_B\right)\right]\right\}$$

$$+ \rho\left\{\epsilon_t - \epsilon(s)_t + p\tau_t + \frac{q_A}{\rho J}\left(\bar{R}_\ell \cdot u_k\right) + \frac{q_B}{\rho J}\left(-\bar{R}_k \cdot u_\ell\right)\right\} = 0.$$

Rearranging and using

$$\rho\tau_t = \frac{\tau_t}{\tau} = \frac{J_t}{J} = \nabla \cdot u,$$

one obtains

$$\rho[u \cdot u_t + \epsilon_t - \epsilon(s)_t] = - u \cdot \nabla p - p\nabla \cdot u$$

$$- \frac{u}{J} \cdot \frac{\partial}{\partial k}\left(\hat{R}\bar{R}_\ell q_A\right) - \frac{q_A \hat{R}}{J}\bar{R}_\ell \cdot \frac{\partial u}{\partial k}$$

$$+ \frac{u}{J} \cdot \frac{\partial}{\partial \ell}\left(\hat{R}\bar{R}_k q_B\right) + \frac{q_B \hat{R}}{J}\bar{R}_k \cdot \frac{\partial u}{\partial \ell}.$$

Use

$$- u \cdot \nabla p - p\nabla \cdot u = - \nabla \cdot (pu) = - \frac{1}{J}\left[\frac{\partial}{\partial k}\left(\hat{R}p\,\bar{R}_\ell \cdot u\right) - \frac{\partial}{\partial \ell}\left(\hat{R}p\,\bar{R}_k \cdot u\right)\right]$$

and collect terms:

$$\rho\left\{\frac{1}{2}u^2 + \epsilon - \epsilon(s)\right\}_t = -J^{-1}\left\{\frac{\partial}{\partial k}\left[(p + q_A)\,u \cdot \left(\hat{R}\,\bar{R}_\ell\right)\right] - \frac{\partial}{\partial \ell}\left[(p + q_B)\,u \cdot \left(\hat{R}\,\bar{R}_k\right)\right]\right\}.$$

Integrate over a radian slice in cylindrical coordinates or over a unit thick slab in Cartesian coordinates.

$$\int_{K_1}^{K_2}\int_{L_1}^{L_2}\left\{\frac{1}{2}u^2 + \epsilon - \epsilon(s)\right\}_t \rho J\, dk\, d\ell = - \int_{K_1}^{K_2}\int_{L_1}^{L_2}\left\{\frac{\partial}{\partial k}\left[(p + q_A)\, u \cdot \left(\hat{R}\,\bar{R}_\ell\right)\right]\right.$$

$$\left. - \frac{\partial}{\partial \ell}\left[(p + q_B)\, u \cdot \left(\hat{R}\,\bar{R}_k\right)\right]\right\} dk\, d\ell. \quad (3.5)$$

This is the desired total energy conservation expression. Note that q_A and q_B appear only in integrals over the boundaries of the system as must be required for a satisfactory artificial viscosity.

Integrated Momentum Equation

$$\frac{d}{dt}\int M u\, dk\, d\ell = \int J\rho u_t\, dk\, d\ell = - \int J\nabla p\, dk\, d\ell - \int \frac{\partial}{\partial k}\left(\hat{R}\,\bar{R}_\ell\, q_A\right) dk\, d\ell$$

$$+ \int \frac{\partial}{\partial \ell}\left(\hat{R}\,\bar{R}_k q_B\right) dk\, d\ell. \quad (3.6)$$

It is seen that q_A and q_B will again only appear in integrals over the boundary of the system. This would not be true for a scalar q in the case of cylindrical coordinates.[3]

Angular Momentum Equation

$$\frac{d}{dt}\int \underset{\sim}{R} \times M\underset{\sim}{u}\, dk\, d\ell = \int \underset{\sim}{R} \times J\rho \underset{\sim}{u}_t\, dk\, d\ell$$

$$= -\int\left\{\underset{\sim}{R} \times (J\nabla p) - \underset{\sim}{R} \times \frac{\partial}{\partial k}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell q_A\right) + \underset{\sim}{R} \times \frac{\partial}{\partial \ell}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k q_B\right)\right\} dk\, d\ell$$

$$= -\int \underset{\sim}{R} \times J\nabla p\, dk\, d\ell - \int \frac{\partial}{\partial k}\left(\underset{\sim}{R} \times \overline{\underset{\sim}{R}}_\ell \hat{R} q_A\right)\, dk\, d\ell$$

$$+ \int \frac{\partial}{\partial \ell}\left(\underset{\sim}{R} \times \overline{\underset{\sim}{R}}_k \hat{R} q_B\right) dk\, d\ell + \int \underset{\sim}{R}_k \times \overline{\underset{\sim}{R}}_\ell \hat{R}(q_A - q_B)\, dk\, d\ell . \tag{3.7}$$

For angular momentum things do not turn out so well. There is a term containing $(q_A - q_B)$ which is integrated over the volume of the system. This will be commented on in the next section.

IV. ARTIFICIAL VISCOSITY

In order to integrate the hydrodynamic equations by replacing them with finite difference equations, a mechanism must be introduced to smooth out the discontinuities which occur when shocks are present. John von Neumann and R. D. Richtmyer[4] first solved this problem in one dimension by introducing an artificial viscosity which spread shock discontinuities out over a specified number of zones.

Essentially they constructed an artificial medium whose final behavior, after experiencing smoothed-out shock waves, was sufficiently like that of the true medium that it could be used in physical equations in place of the true medium.

This artificial medium was defined as follows: Consider a plane shock in the R direction only, no motion in the Z direction. Under these circumstances the Lagrangian coordinates can be chosen orthogonal with k and ℓ increasing in the R and Z directions, respectively, i. e.,

$$R_\ell = Z_k = 0.$$

Assume also a γ-law gas equation of state,

$$p = \epsilon(\gamma - 1)/\tau.$$

Then, in both the momentum and energy equation, p is replaced by $(p + q_N)$, where q_N is defined as follows. Let

$$\left(\frac{\partial u}{\partial k}\right)_- = \min\left[\frac{\partial u}{\partial k}, 0\right] \text{ and } c_0 = \frac{N}{\pi}\sqrt{\frac{\gamma+1}{2}},$$

then

$$q_N = c_0^2 \rho \left[\left(\frac{\partial u}{\partial k}\right)_-\right]^2 .$$

The quantity N appearing in the expression for c_0 is the distance the shock will be spread over in k space in a Cartesian system ($\Delta k = N$ from beginning of shock to the end). This is all that will be said about the Richtmyer-von Neumann paper.

The following is a list of some physical conditions one would like an artificial viscosity to satisfy:

a) A uniform expansion or contraction over the entire medium is considered to correspond to a reversible process (infinitesimal volumes are collapsed with infinitesimal slowness). The artificial viscosity should be independent of such motion.[5]

b) The velocity component (of the medium) parallel to the shock front should be continuous.

c) Angular momentum should be conserved.

d) There should be no artificial viscosity under a velocity field corresponding to a rigid rotation.

The Richtmyer-von Neumann one-dimensional viscosity does not satisfy condition (a). This is the case where

$$\frac{\partial u}{\partial R} < 0, \quad \frac{\partial^2 u}{\partial R^2} = 0, \quad \frac{\partial \tau}{\partial R} = 0,$$

then

$$q_N = c_0^2 \left(\frac{\partial u}{\partial k}\right)^2 \neq 0.$$

In place of this basic viscosity, let us use the following. Define

$$q = - c_0^2 \rho \left(\frac{\partial u}{\partial k}\right)_- \cdot \left| \frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k}\right)_- \right| .$$

For the case of uniform zones this q will vanish as desired for the situation described by condition (a).[6]

In practice one chooses $c_0^2 = 2$. For q_N, this spreads a simple shock out over three to four zones. That is, $\left(\frac{\partial u}{\partial k}\right)_-$ is nonvanishing only over three to four zones. While $\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k}\right)_-$ may be somewhat smaller in magnitude than $\left(\frac{\partial u}{\partial k}\right)_-$, it should be of the same order of magnitude. Thus we may hope that q, with the same c_0^2, will also spread a simple shock out over three to four zones. In actual practice this turns out to be the case. The shape of the shock front of course departs from the simple sine wave of the Richtmyer-von Neumann q_N. The tensor viscosity to be developed in this paper will thus be designed around this new scalar viscosity.

Consider now a region in a general k, ℓ space. $\underset{\sim}{R}_k$ and $\underset{\sim}{R}_\ell$ are vectors pointing along lines of constant ℓ and k, respectively. More significant directions, however, are those given by the normal vectors:

$$\underset{\sim}{\overline{R}}_k = (Z_k, -R_k) = \text{normal vector to } \underset{\sim}{R}_k,$$

$$\underset{\sim}{\overline{R}}_\ell = (Z_\ell, -R_\ell) = \text{normal vector to } \underset{\sim}{R}_\ell.$$

These normal vectors are the ones that appear in the $\underset{\sim}{D}$ operator. A scalar viscosity appears in the momentum equation acted on by this operator.

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{D} p = \frac{-1}{J} \underset{\sim}{\overline{R}}_\ell \frac{\partial q}{\partial k} + \frac{1}{J} \underset{\sim}{\overline{R}}_k \frac{\partial q}{\partial \ell}.$$

Condition (b), however, requires directional properties. As a first attempt at a tensor equation we can write

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{D} p = \frac{-1}{J} \underset{\sim}{\overline{R}}_\ell \frac{\partial q_A}{\partial k} + \frac{1}{J} \underset{\sim}{\overline{R}}_k \frac{\partial q_B}{\partial \ell},$$

where q_A is a "one-dimensional" viscosity associated with shocks traveling in the $\underset{\sim}{\overline{R}}_\ell$ direction and analogously for q_B. An internal energy equation can be derived through the requirement of total energy conservation, that is, require that (3.5) be valid, then one obtains

$$\frac{\partial \epsilon}{\partial t} - \frac{\partial \epsilon(s)}{\partial t} + p \frac{\partial \tau}{\partial t} = - \frac{q_A}{M} \frac{\partial}{\partial k}\left(\hat{R}\, \underset{\sim}{\overline{R}}_\ell \cdot \underset{\sim}{u}\right) + \frac{q_B}{M} \frac{\partial}{\partial \ell}\left(\hat{R}\, \underset{\sim}{\overline{R}}_k \cdot \underset{\sim}{u}\right).$$

This is not a satisfactory energy equation since it is not Galilean invariant. To make it independent of a uniform velocity transformation we must take $\hat{R}\, \underset{\sim}{\overline{R}}_\ell$ and $\hat{R}\, \underset{\sim}{\overline{R}}_k$ outside of the derivatives over k and ℓ, respectively. Then

$$\frac{\partial \epsilon}{\partial t} - \frac{\partial \epsilon(s)}{\partial t} + p \frac{\partial \tau}{\partial t} = - \frac{q_A}{\rho J}(\underset{\sim}{\overline{R}}_\ell \cdot \underset{\sim}{u}_k) - \frac{q_B}{\rho J}(-\underset{\sim}{\overline{R}}_k \cdot \underset{\sim}{u}_\ell).$$

Going back through the total energy conservation equation once more, there results

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \frac{\nabla p}{\rho} = -\frac{1}{M}\left[\frac{\partial}{\partial k}\left(\hat{R}\, \underset{\sim}{\overline{R}}_\ell q_A\right) - \frac{\partial}{\partial \ell}\left(\hat{R}\, \underset{\sim}{\overline{R}}_k q_B\right)\right],$$

and we have the equations (3.2) and (3.3).

There remains only the problem of finding suitable expressions for q_A and q_B. For a shock driven in the $\underset{\sim}{\overline{R}}_\ell$ direction the previous one-dimensional viscosity can be generalized as follows. Define

$$\left(\frac{\partial u}{\partial k}\right)^A = \frac{\underset{\sim}{R}_\ell \cdot \underset{\sim}{u}_k}{|\overline{\underset{\sim}{R}}_\ell|}$$

and also

$$\left(\frac{\partial u}{\partial k}\right)^A_- = \min\left\{\left(\frac{\partial u}{\partial k}\right)^A, 0\right\},$$

then let

$$q_A = -2\rho\left(\frac{\partial u}{\partial k}\right)^A_- \cdot \left|\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k}\right)^A_-\right|. \tag{4.1}$$

Analogously,

$$\left(\frac{\partial u}{\partial \ell}\right)^B = \frac{\left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell\right)}{|\overline{\underset{\sim}{R}}_k|},$$

$$\left(\frac{\partial u}{\partial \ell}\right)^B_- = \min\left\{\left(\frac{\partial u}{\partial \ell}\right)^B, 0\right\},$$

$$q_B = -2\rho\left(\frac{\partial u}{\partial \ell}\right)^B_- \cdot \left|\frac{\partial}{\partial \ell}\left(\frac{\partial u}{\partial \ell}\right)^B_-\right|. \tag{4.2}$$

The definition of q_A and q_B provided by equations (4.1) and (4.2) completes the definition of the tensor artificial viscosity.[7] The equations (3.1), (3.2), (3.3), (3.4), (4.1), and (4.2) together with specified boundary and initial conditions determine the state of the system throughout time.

There is a general rule in computing that you can always find a situation which your code will not handle. Consider the physical conditions (c) and (d) listed earlier. They were chosen because they are conditions that the specified tensor viscosity does not satisfy. The development at the end of the last section shows that angular momentum is not conserved in general. Without going into details, the difficulty seems to lie in the fact that most problems of

interest contain a great many long thin zones. **The artificial viscosity used must have the property of spreading shocks over a small distance in the direction of fine zoning and over a large distance in the direction of gross zoning, i. e., the viscosity is a zone-dependent quantity. This seems to preclude angular momentum conservation.**

This is not a problem in practice since Lagrangian codes are not suitable for problems with turbulent flow. For these problems one should use an Eulerian code and here there is no problem in producing angular momentum conservation. One can see this in a rough way by noting that for an orthogonal grid, $\underset{\sim}{R}_k$ and $\overline{\underset{\sim}{R}}_\ell$ are parallel and thus the undesired term will vanish from equation (3.7).

Condition (d) refers to a velocity field of the type

$$\underset{\sim}{u} = \overline{\underset{\sim}{R}},$$

for which $\left(\frac{\partial u}{\partial k}\right)^A$ and $\left(\frac{\partial u}{\partial \ell}\right)^B$ do not vanish in general. This difficulty could possibly be fixed up but since it seems related to the problem of angular momentum conservation, it did not seem worth while to fix this one up and not be able to do anything about the other.

Tensor viscosities are new quantities in numerical hydrodynamics and we will now consider them from the Eulerian viewpoint for a moment. This will demonstrate explicitly the tensor property of the defined viscosity and other matters of general interest. (In a strict sense, however, the following discussion is redundant as far as this paper is concerned.)

In Eulerian space we can define a momentum equation as follows:

$$\rho\frac{\partial u^i}{\partial t} + \frac{\partial p}{\partial x^i} + \frac{\partial Q^{ji}}{\partial x^j} = 0. \tag{4.3}$$

In analogous fashion to the Lagrangian case, the requirement of total energy conservation then specifies the form of the internal energy equation.

$$\frac{\partial \epsilon}{\partial t} + P\frac{\partial \tau}{\partial t} + \frac{Q^{ji}}{\rho}\frac{\partial u^i}{\partial x^j} \tag{4.4}$$

Let us reduce these three-dimensional Cartesian space equations to one-dimensional ones for Cartesian, cylindrical, and spherical coordinates.

Using the machinery of tensor analysis, let us write the tensor viscosity equations in covariant form. Let $t^{ab\cdots}$ be an arbitrary contravariant tensor and $\xi^\ell, \ell = 1$ to 3, be an arbitrary coordinate system. The covariant derivative of $t^{ab\cdots}$ is

$$t^{ab\cdots}{}_{;\ell} = \frac{\partial t^{ab\cdots}}{\partial \xi^\ell} + \left\{ {a \atop p\ell} \right\} t^{pb\cdots} + \left\{ {b \atop p\ell} \right\} t^{ap\cdots} + \cdots$$

where

$$\left\{ {\ell \atop ik} \right\} = \tfrac{1}{2} g^{\ell s}\left(g_{is,k} + g_{ks,i} - g_{ik,s}\right),$$

$$ds^2 = g_{i\ell}\, d\xi^i d\xi^\ell = \text{invariant length.}$$

Also let

$$\frac{\partial t^{ab\cdots}}{\partial t} = \left.\frac{\partial t^{ab\cdots}}{\partial t}\right|_{\xi^\ell \text{ fixed}} + u^j t^{ab\cdots}{}_{;j}\,.$$

The covariant tensor viscosity equations are

$$\rho\frac{\partial u^i}{\partial t} = -g^{ij}P_{;j} - Q^{ji}{}_{;j}\,, \tag{4.5}$$

$$\frac{\partial \xi}{\partial t} = -P\frac{\partial \tau}{\partial t} - \frac{Q^{ji}}{\rho} g_{i\ell} u^\ell{}_{;j}\,. \tag{4.6}$$

For cylindrical coordinates

$$\xi^1 = R, \qquad \xi^2 = Z, \qquad \xi^3 = \theta,$$
$$g_{11} = g_{22} = 1, g_{33} = R^2, \qquad g^{ii} = g_{ii}^{-1}, \qquad g_{ij} = g^{ij} = 0 \quad \text{if } i \neq j.$$

For spherical coordinates

$$\xi^1 = R, \qquad \xi^2 = \phi, \qquad \xi^3 = \theta,$$
$$g_{11} = 1, \quad g_{22} = R^2 \sin^2\theta, \quad g_{33} = R^2, \quad g^{ii} = g_{ii}^{-1},$$
$$g_{ij} = g^{ij} = 0 \qquad \text{if } i \neq j.$$

By turning the crank we can now obtain the desired one-dimensional equations. Define

$$\delta = \begin{cases} 0 & \text{Cartesian,} \\ 1 & \text{cylindrical,} \\ 2 & \text{spherical,} \end{cases}$$

then the result is

(One-Dimensional Tensor Viscosity Equations)

$$\rho\frac{\partial u^R}{\partial t} = -\frac{\partial p}{\partial R} - \frac{1}{R^\delta}\frac{\partial}{\partial R}\left(R^\delta Q^{RR}\right), \tag{4.7}$$

$$\frac{\partial \epsilon}{\partial t} = -P\frac{\partial \tau}{\partial t} - \frac{Q^{RR}}{\rho}\frac{\partial u^R}{\partial R}\,. \tag{4.8}$$

Q^{RR} is all that remains of the tensor Q^{ij}. Its value in the given coordinate system is to be taken the same as that of the scalar q defined earlier.

By integrating the momentum equation (4.7), we can see that even in one dimension a tensor viscosity has advantages.

$$\int_{R_1}^{R_2} \rho\frac{\partial u^R}{\partial t} R^\delta dR = -\int_{R_1}^{R_2} \frac{\partial p}{\partial R} R^\delta dR - R^\delta Q^{RR}\Big|_{R_1}^{R_2}.$$

If Q^{RR} vanishes at the boundary points R_1 and R_2, then the artificial viscosity vanishes completely from the equation. In short we have the proper

expression for the sum of the "radial momentum" in the cylindrical and spherical case. A scalar viscosity will only produce the proper expression for the Cartesian case.

A second advantage appears in the energy equation. The entropy of the system is changed by multiplying Q^{RR} by $\frac{\partial u^R}{\partial R}$ instead of $\frac{\partial \tau}{\partial t}$. $\frac{\partial u^R}{\partial R}$ is a function only of the compression of the system in the radial direction. $\frac{\partial \tau}{\partial t}$ will in general also contain contributions from the general radial convergence or divergence of the system as it moves inward or outward.

The two-dimensional Eulerian equations for a Cartesian-cylindrical system, i.e., the Eulerian equations corresponding to (3.2) and (3.3), can also be deduced from the general covariant equations. They are

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial R} - \frac{1}{\hat{R}} \frac{\partial}{\partial R}(\hat{R} Q^{RR}) - \frac{\partial Q^{ZR}}{\partial Z},$$

$$\rho \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial Z} - \frac{1}{\hat{R}} \frac{\partial}{\partial R}(\hat{R} Q^{RZ}) - \frac{\partial Q^{ZZ}}{\partial Z},$$

$$\frac{\partial \epsilon}{\partial t} = - P \frac{\partial \tau}{\partial t} - \frac{1}{\rho}\left(Q^{RR} \frac{\partial u}{\partial R} + Q^{ZR} \frac{\partial u}{\partial Z} + Q^{RZ} \frac{\partial v}{\partial R} + Q^{ZZ} \frac{\partial v}{\partial Z}\right).$$

By transforming these equations to Lagrangian k, ℓ space and equating to the equations (3.2) and (3.3), we obtain the following expressions for the elements of the tensor.

$$Q^{RR} = \frac{1}{J}\left[\frac{\partial R}{\partial k} \frac{\partial Z}{\partial \ell} q_A - \frac{\partial R}{\partial \ell} \frac{\partial Z}{\partial k} q_B\right],$$

$$Q^{ZR} = \frac{1}{J}\left[\frac{\partial Z}{\partial k} \frac{\partial Z}{\partial \ell}(q_A - q_B)\right],$$

$$Q^{RZ} = \frac{1}{J}\left[\frac{\partial R}{\partial k} \frac{\partial R}{\partial \ell}(q_B - q_A)\right],$$

$$Q^{ZZ} = \frac{1}{J}\left[\frac{\partial R}{\partial k} \frac{\partial Z}{\partial \ell} q_B - \frac{\partial R}{\partial \ell} \frac{\partial Z}{\partial k} q_A\right].$$

This explicitly demonstrates the tensor character of the artificial viscosity. One may note that the tensor is not symmetric. This is in confirmation of the previously observed failure of angular momentum conservation.

V. STABILITY OF THE DIFFERENTIAL EQUATIONS

The dominant behavior of the system under short-wavelength perturbations is examined for two different cases. First, one considers non-shock hydrodynamics alone. Second, hydrodynamics when shocks are present. The introduction of shocks leads to a dominant behavior which is both different and higher in order than the previous simpler case. Since both of these cases will occur in general during the course of a problem, the requirement of stable behavior for both is an obvious necessity.

While it may be plausible it is certainly not obvious that the simultaneous satisfaction of the requirements resulting from these two cases is also sufficient to insure stability of the complete system. In practice, however, this turns out to be the case.

In the following development $\epsilon(s)$ is ignored since its variation vanishes and thus it can never contribute to the final results.

Case I: Non-Shock Hydrodynamics

The absence of artificial viscosity leaves us with the following differential equations.

$$\underline{u}_t + \frac{1}{\rho} \nabla p = 0,$$

$$\epsilon_t + p\tau_t = 0.$$

The internal energy equation is equivalent to the statement that the entropy is a constant in time,

$$S_t = 0,$$

thus we will choose

$$p = p(S, \rho)$$

and define

$$c^2 = \frac{\partial p}{\partial \rho}\bigg|_S = \text{sound speed squared}$$

as is usually done. The momentum equation can now be written

$$\underset{\sim}{u}_t + \frac{c^2}{\rho}\nabla\rho + \frac{1}{\rho}\frac{\partial p}{\partial S}\bigg|_\rho \nabla S = 0.$$

Using

$$\frac{1}{\rho}\nabla\rho = \frac{-1}{\tau}\nabla\tau,$$

$$\nabla = \frac{1}{\underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell}\left[\overline{\underset{\sim}{R}}_\ell \frac{\partial}{\partial k} - \overline{\underset{\sim}{R}}_k \frac{\partial}{\partial \ell}\right],$$

we obtain

$$\underset{\sim}{R}_{tt} = \frac{c^2}{\underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell \tau}[\overline{\underset{\sim}{R}}_\ell \tau_k - \overline{\underset{\sim}{R}}_k \tau_\ell] - \frac{\tau \frac{\partial p}{\partial S}\big|_\rho}{\underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell}[\overline{\underset{\sim}{R}}_\ell S_k - \overline{\underset{\sim}{R}}_k S_\ell] \tag{5.1}$$

where

$$\tau = \frac{\hat{R}}{M}[\underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell]. \tag{5.2}$$

The Eqs. (5.1) and (5.2) comprise three equations in three unknowns, $\underset{\sim}{R}$ and τ. M and S are known functions of k and ℓ. c^2 and $\partial p/\partial S$ are known functions of τ and S.

During the numerical integration of these equations, the principal type of error that will be introduced into the solution corresponds to a small perturbation, localized in a small region of space. Following von Neumann and Richtmyer, equations describing the future behavior of this disturbance are produced by taking a first variation.

$$\underset{\sim}{R} \to \underset{\sim}{R} + \delta\underset{\sim}{R},$$

$$\tau \to \tau + \delta\tau.$$

$\underset{\sim}{R}$, τ, M, S, c^2, and $\frac{\partial p}{\partial S}\big|_\rho$ which appear in the resulting equations are assumed to vary slowly over the region of the perturbation, so that they can be considered as constants. We then have linear equations and need only examine the behavior of various Fourier components of the disturbance and, in particular, a solution of the form

$$\delta\underset{\sim}{R} = \delta\underset{\sim}{R}_0 \exp(im_1 k + im_2 \ell + \alpha t), \quad \delta\tau = \delta\tau_0 \exp(im_1 k + im_2 \ell + \alpha t),$$

is assumed. Furthermore, due to the localization in a small region, short wavelengths will comprise the dominant modes, allowing m_1 and m_2 to be considered as large quantities.

Consider, for a moment, cylindrical coordinates only. Then (5.2) gives

$$\delta\tau = \frac{\delta R}{M}[\underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell] + \frac{R}{M}[\overline{\underset{\sim}{R}}_\ell \cdot \delta\underset{\sim}{R}_k - \overline{\underset{\sim}{R}}_k \cdot \delta\underset{\sim}{R}_\ell].$$

Only terms of highest order in m, n will be kept. Any derivative over k and ℓ will bring down one power of m_1 or m_2. Thus δR will be dropped in the above equation as it is of zero order while $\delta\underset{\sim}{R}_k$ and $\delta\underset{\sim}{R}_\ell$ are of first order. We thus obtain

$$\delta\tau = \tau\underset{\sim}{D} \cdot \delta\underset{\sim}{R}. \tag{5.3}$$

Note at this point the Eq. (5.3) holds for both Cartesian and cylindrical coordinates, i. e., the main difference between the two cases appeared in the lower order term which was dropped.

With the observation from Eq. (5.3) that $\delta\tau$ should be weighted one order higher than $\delta\underset{\sim}{R}$, one may now vary Eq. (5.1) in the same manner. The result is

$$\delta \underset{\sim}{R}_{tt} = \frac{c^2}{\tau} \underset{\sim}{D} \delta\tau. \tag{5.4}$$

Combining (5.3) and (5.4) one obtains

$$\delta\tau_{tt} = c^2 \underset{\sim}{D} \cdot \underset{\sim}{D} \delta\tau, \tag{5.5}$$

the effective sound equation for our system. Substituting the assumed $\delta\tau$ in (5.5) gives

$$a^2 = -\frac{c^2}{j^2} [\underset{\sim}{R}_\ell m_1 - \underset{\sim}{R}_k m_2]^2.$$

Thus a is an imaginary, as is to be expected for the sound equation. Since perturbations merely oscillate sinusoidally and do not grow in time, the differential equations of Case I are stable under this type of disturbance, provided the various assumptions made in the analysis are reasonably valid, as they usually seem to be in practice.[8]

Case II: Shock Region Hydrodynamics

Here one considers the case when the artificial viscosity is present. We will first go through the development for a simple scalar viscosity and then discuss slightly the more complicated tensor viscosity case. These scalar viscosity equations are ones that were used in earlier versions of the code and the stability requirement they produce also has the property of being a reasonable accuracy-of-integration criterion. Thus it seemed worthwhile to keep this criterion in the code even though it had been deduced for a different set of equations.

Define

$$\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k\right)_- = \min\left[\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k\right), 0\right],$$

$$\left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell\right)_- = \min\left[\left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell\right), 0\right],$$

then the first system under consideration can be written as follows.

Scalar Viscosity Hydrodynamics

$$\underset{\sim}{u}_t + \tau \underset{\sim}{D}(p+q) = 0, \qquad \epsilon_t = -(p+q)\tau_t,$$

$$p = p(\tau, \epsilon), \qquad \tau = \frac{\hat{R}}{M} \underset{\sim}{R}_k \cdot \overline{\underset{\sim}{R}}_\ell,$$

$$q = c_0^2 \rho \left[\frac{\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k\right)_-^2}{\underset{\sim}{R}_\ell^2} + \frac{\left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell\right)_-^2}{\underset{\sim}{R}_k^2}\right].$$

The next step is to carry out the variation, dropping all but the highest order terms. Consider the relative weights of δp and δq.

Use $\delta \underset{\sim}{R}$ as a reference point of zero weight.

$$\delta q = c_0^2 \rho \frac{2\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k\right)_-}{\underset{\sim}{R}_\ell^2} \overline{\underset{\sim}{R}}_\ell \cdot \delta \underset{\sim}{u}_k + \ldots .$$

Since

$$\delta \underset{\sim}{u}_k = \delta \underset{\sim}{R}_{kt}$$

it is seen that δq is at least of first order in m_1, m_2 and of first order in a.

$$\delta p = \left.\frac{\partial p}{\partial \tau}\right|_\epsilon \delta\tau + \left.\frac{\partial p}{\partial \epsilon}\right|_\tau \delta\epsilon.$$

Substitute

$$\delta\epsilon = -(p+q)\,\delta\tau.$$

Then

$$\delta p = \left[\left.\frac{\partial p}{\partial \tau}\right|_\epsilon - \left.\frac{\partial p}{\partial \epsilon}\right|_\tau (p+q)\right] \delta\tau.$$

Thus δp is of the same order as $\delta\tau$. From Case I it is seen that $\delta\tau$ is first order in m_1, m_2 but zero in a. So δp is one order lower in a than δq and may be dropped in comparison. Proceeding in this manner,[9] one is finally left with

$$\delta \underset{\sim}{u}_t = -\tau \underset{\sim}{D} \delta q,$$

$$\delta q = A \overline{\underset{\sim}{R}}_\ell \cdot \delta \underset{\sim}{u}_k - B \overline{\underset{\sim}{R}}_k \cdot \delta \underset{\sim}{u}_\ell ,$$

where

$$A = \frac{2 c_0^2 \rho}{\underset{\sim}{R}_\ell^2} \left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k \right)_- ,$$

$$B = \frac{2 c_0^2 \rho}{\underset{\sim}{R}_k^2} \left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell \right)_- .$$

Combining the previous equations gives

$$\delta q_t = -\tau \left[A \overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{D} \delta q_k - B \overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{D} \delta q_\ell \right];$$

using the definition of $\underset{\sim}{D}$ (in Section II) gives

$$\delta q_t = -\frac{\tau}{j} \left[A \underset{\sim}{R}_\ell^2 \, \delta q_{kk} - (A+B) \underset{\sim}{R}_k \cdot \underset{\sim}{R}_\ell \, \delta q_{\ell k} + B \underset{\sim}{R}_k^2 \, \delta q_{\ell \ell} \right]. \quad (5.6)$$

Let

$$\delta q = \delta q_0 \exp(i m_1 k + i m_2 \ell + \alpha t)$$

and use the values of A and B for q; then

$$\alpha = +\frac{2 c_0^2}{j} \left[\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k \right)_- m_1^2 + \left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell \right)_- m_2^2 - \underset{\sim}{R}_k \cdot \underset{\sim}{R}_\ell \left(\frac{\left(\overline{\underset{\sim}{R}}_\ell \cdot \underset{\sim}{u}_k \right)_-}{\underset{\sim}{R}_\ell^2} + \frac{\left(-\overline{\underset{\sim}{R}}_k \cdot \underset{\sim}{u}_\ell \right)_-}{\underset{\sim}{R}_k^2} \right) m_1 m_2 \right].$$

For some situations the value of α may be plus. Thus these differential equations are not always stable under short-wavelength perturbations. However, it will be shown later that the difference equations are stable, conditionally, in all cases (to the extent that one can believe this analysis, anyway). More to the point, when this system is differenced and put on a computer it produces a fairly reliable code.

Tensor Viscosity Hydrodynamics

Things do not work out so nicely for this case. The thing which complicates the situation is really not the tensor property but the fact that we used a new "one-dimensional" viscosity for q_A and q_B instead of generalizations of the old Richtmyer-von Neumann expression. It is the factor analogous to

$$\left| \frac{\partial}{\partial k} \left(\frac{\partial u}{\partial k} \right)_- \right|$$

that prevents one from doing a completely parallel development for the tensor case. All this could possibly be worked out but at this time it doesn't seem worth while. In the first place the criterion deduced originally from the scalar equations and now considered as an "accuracy criterion" is sufficient to insure stability of the tensor equations as demonstrated by many hours of production runs. In the second place, the first place is all that counts, i.e., any criterion deduced from rough analysis like this would have to be tested on a computer anyway, so why bother any further?

VI. HYDRODYNAMIC DIFFERENCE EQUATIONS

Up to this point a k or ℓ subscript on a variable has stood only for partial derivatives with respect to that variable. When a comma is introduced between k and ℓ, the k,ℓ subscript will stand for discrete k,ℓ variables. Thus

$$\underset{\sim}{R}_{k\ell} = \frac{\partial^2 R}{\partial k \partial \ell}$$

while

$$\underset{\sim}{R}^n_{k,\ell} = \underset{\sim}{R}(k, \ell, t^n), \quad k, \ell, n = 0, 1, 2, \ldots ,$$

where n is a discrete time variable.

$$t^n = t^{n-1} + Dt^{n-1/2} = \sum_{s=1}^{n} Dt^{s-1/2}.$$

A typical zone in discrete k, ℓ space is represented in Fig. 2. Straightforward differencing leads one to define and calculate thermodynamic quantities such as p, q_A, and q_B at the center of the quadrilateral zone. However, the hydrodynamic motion of material, using this type of differencing, is not always as stable as one would like. Consider long thin zones being collapsed on one side (Fig. 3a). Since the left side of the center zone is being collapsed, a q must be present to resist this motion and to raise the entropy. However a q calculated at the center of the zone will also move apart the mesh points on the right side of the center zone. This is a completely fallacious effect and it leads to undesirable drifting of mesh points in essentially cold regions of the problem.

The previous trouble occurs even in cases of smooth hydrodynamic motion. This is the type of problem for which Lagrangian difference equations are most suitable. However, the presence of just a small amount of turbulence will also lead to irregular or so-called "pathological zones" of the type shown in Fig. 3b. Zones of this type were observed early in the game. The difference equations in use then did not move the medium properly when these zones were present. This led to the development of "triangular hydrodynamics."

In the most straightforward triangular system one merely draws one of the diagonals of the quadrilateral and defines mass conservation separately within the two triangular zones (Fig. 4). The quantities ϵ, p, and q are now defined at the centers of the triangular zones. These zones cannot turn inside out without their specific volume vanishing. The pressure would go

through infinity in this case, i. e., pathological zones cannot occur. Triangular zones also solve the problem of the drifting of points due to the propagation of a q disturbance. Successful triangular hydrodynamic codes have been written and are now in use.

The triangles serve to introduce some constraints into the system. These constraints are essentially nonphysical. An element of fluid, initially triangular in shape, is required to maintain a triangular shape throughout all time.

A quadrilateral code requires a volume element to remain a four-sided figure. Thus it also constrains the medium, but to a lesser extent than a three-sided figure. In this work an attempt was made to write quadrilateral difference equations which behaved reasonably correctly for "pathological zones" and thus eliminate the need for additional constraints.

The difference equations in this paper use a quadrilateral zone (Fig. 5a) with a p and ϵ defined at its center and with four q's defined off toward the sides of the zone.[10] These four q's act to stabilize zone shape. For example, in the case of a regular zone collapsing on one side, the four q's per zone are calculated so as to give the effect shown in Fig. 5b, thus eliminating the propagation of a q disturbance.

Some notation will be introduced at this point. Differences over the k variable will be represented by Δ; for example,

$$\Delta \underset{\sim}{R}^n_{k+\frac{1}{2},\ell} = \underset{\sim}{R}^n_{k+1,\ell} - \underset{\sim}{R}^n_{k,\ell}.$$

Similarly we will use

$$\delta \underset{\sim}{R}^n_{k,\ell+\frac{1}{2}} = \underset{\sim}{R}^n_{k,\ell+1} - \underset{\sim}{R}^n_{k,\ell}.$$

It follows directly that

$$\Delta \underset{\sim}{\overline{R}}^{n}_{k+\frac{1}{2},\ell} = \left(\Delta Z^{n}_{k+\frac{1}{2},\ell}\,,\ -\Delta R^{n}_{k+\frac{1}{2},\ell}\right),$$

$$\delta \underset{\sim}{\overline{R}}^{n}_{k,\ell+\frac{1}{2}} = \left(\delta Z^{n}_{k,\ell+\frac{1}{2}}\,,\ -\delta R^{n}_{k,\ell+\frac{1}{2}}\right),$$

in identical fashion to the differential case. Differences over the time variable n will be represented by D, for example,

$$D\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \tau^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} - \tau^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

On basic quantities such as R or τ, simple averages will be indicated by subscript values, for example

$$\Delta \underset{\sim}{R}^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{1}{2}\left[\Delta \underset{\sim}{R}^{n}_{k-\frac{1}{2},\ell} + \Delta \underset{\sim}{R}^{n}_{k-\frac{1}{2},\ell-1}\right].$$

Details of how a two-dimensional hydrodynamic code is programmed will be ignored in this paper. We will assume that the following quantities are known at time t^n for all k and ℓ.

T Storage

$$M_{k-\frac{1}{2},\ell-\frac{1}{2}},\quad P^{n-1}_{k-\frac{1}{2},\ell-\frac{1}{2}},\quad u^{n-\frac{1}{2}}_{k,\ell},\quad v^{n-\frac{1}{2}}_{k,\ell},\quad R^{n}_{k,\ell},$$

$$Z^{n}_{k,\ell},\quad j^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},\quad \tau^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},\quad {}_1q^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},\quad {}_2q^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$${}_3q^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}\quad {}_4q^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}\quad \epsilon^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}.$$

The difference equations presented in this section will advance these quantities from n to $n+1$. It will further be assumed that their values at time n will always be available to us during this process. These quantities are not all independent, as can be seen from the difference equations which will follow shortly. They form a convenient set, however.

In addition we need

$$Dt^{n-\frac{1}{2}},\qquad Dt^{n},\qquad Dt^{n+\frac{1}{2}}.$$

These are related by

$$Dt^{n} = \frac{1}{2}\left[Dt^{n-\frac{1}{2}} + Dt^{n+\frac{1}{2}}\right].$$

The calculation of $Dt^{n+3/2}$ will be considered in the next section.

The first quantity in T storage, $M_{k-\frac{1}{2},\ell-\frac{1}{2}}$, is the mass of a zone. This is a constant in time, calculated by the generator and never changed thereafter.

$$M_{k-\frac{1}{2},\ell-\frac{1}{2}} = \rho^{0}_{k-\frac{1}{2},\ell-\frac{1}{2}} J^{0}_{k-\frac{1}{2},\ell-\frac{1}{2}}. \tag{6.1}$$

The calculation of the volume of a zone, $J^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}$, will be described later on in this section.

From a given function or table one evaluates $P^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}$.

$$P^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} = P\left(\epsilon^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}},\ \tau^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right). \tag{6.2}$$

P^n is extrapolated or interpolated to a "centered time n" for use in the momentum equation.

$$P^{cn}_{k-\frac{1}{2},\ell-\frac{1}{2}} = P^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} + \left(\frac{Dt^{n+\frac{1}{2}} - Dt^{n-\frac{1}{2}}}{4}\right)\left(\frac{P^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} - P^{n-1}_{k-\frac{1}{2},\ell-\frac{1}{2}}}{Dt^{n-\frac{1}{2}}}\right).$$

This "centered time n" is the midpoint between the times $n-\frac{1}{2}$ and $n+\frac{1}{2}$ (Fig. 6). This centering is of use only when the time increment $Dt^{n+\frac{1}{2}}$ is different from that of the previous time cycle. It is not obvious that this centering produces a significant effect. It was easy to do, so it was done. No investigations were made as to how much, if any, improvement it makes in the running of an actual problem.

We define

$$\omega_{k+\frac{1}{2},\ell} = \left[\tfrac{1}{4}\left(\Delta \underset{\sim}{R}^n_{k+\frac{1}{2},\ell+\frac{1}{2}} + \Delta \underset{\sim}{R}^n_{k+\frac{1}{2},\ell-\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} ,$$

$$\omega_{k,\ell+\frac{1}{2}} = \left[\tfrac{1}{4}\left(\delta \underset{\sim}{R}^n_{k+\frac{1}{2},\ell+\frac{1}{2}} + \delta \underset{\sim}{R}^n_{k-\frac{1}{2},\ell+\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} .$$

$$\xi'_{k,\ell} = \frac{2\omega_{k-\frac{1}{2},\ell}}{\omega_{k+\frac{1}{2},\ell}+\omega_{k-\frac{1}{2},\ell}} , \quad \eta'_{k,\ell} = \frac{2\omega_{k,\ell-\frac{1}{2}}}{\omega_{k,\ell+\frac{1}{2}}+\omega_{k,\ell-\frac{1}{2}}} ,$$

$$\xi_{k,\ell} = \max\left\{0.2, \ \min\left[\xi'_{k,\ell}, \ 1.8\right]\right\} ,$$

$$\eta_{k,\ell} = \max\left\{0.2, \ \min\left[\eta'_{k,\ell}, \ 1.8\right]\right\} .$$

ξ and η are a pair of weighting functions:

$$0.2 \le \xi_{k,\ell} \le 1.8, \ 0.2 \le \eta_{k,\ell} \le 1.8 .$$

They are used in the momentum equation to average quantities on the basis of their position in R, Z space. This was found to produce better energy conservation than just taking a simple average in k, ℓ space.

With this much background we can proceed with the differencing of the momentum equation. Proceeding from the general to the particular, i. e., in reverse order of actual calculation, we have,

$$\underset{\sim}{u}^{n+\frac{1}{2}}_{k,\ell} = \underset{\sim}{u}^{n-\frac{1}{2}}_{k,\ell} + D\underset{\sim}{u}^n_{k,\ell} , \qquad (6.3)$$

$$D\underset{\sim}{u}^n_{k,\ell} = -Dt^n\left\{\tau\nabla p + M^{-1}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell q_A\right)_k - M^{-1}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k q_B\right)_\ell\right\}^n_{k,\ell} ,$$

$$\left\{\tau\nabla p + M^{-1}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell q_A\right)_k - M^{-1}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k q_B\right)_\ell\right\}^n_{k,\ell}$$

$$= \eta_{k,\ell}\left[\frac{\overline{\underset{\sim}{R}}_\ell}{2\rho j}\frac{\partial p}{\partial k} + \frac{1}{2M}\frac{\partial}{\partial k}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell \ {}_1q_A\right)\right]^n_{k,\ell-\frac{1}{2}}$$

$$+(2-\eta_{k,\ell})\left[\frac{\overline{\underset{\sim}{R}}_\ell}{2\rho j}\frac{\partial p}{\partial k} + \frac{1}{2M}\frac{\partial}{\partial k}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell \ {}_3q_A\right)\right]^n_{k,\ell+\frac{1}{2}}$$

$$-\xi_{k,\ell}\left[\frac{\overline{\underset{\sim}{R}}_k}{2\rho j}\frac{\partial p}{\partial \ell} + \frac{1}{2M}\frac{\partial}{\partial \ell}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k \ {}_2q_B\right)\right]^n_{k-\frac{1}{2},\ell}$$

$$-(2-\xi_{k,\ell})\left[\frac{\overline{\underset{\sim}{R}}_k}{2\rho j}\frac{\partial p}{\partial \ell} + \frac{1}{2M}\frac{\partial}{\partial \ell}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k \ {}_4q_B\right)\right]^n_{k+\frac{1}{2},\ell} .$$

Note that in the last equation we have differentiated between ${}_1q_A$ and ${}_3q_A$ and between ${}_2q_B$ and ${}_4q_B$. The four square-bracket expressions correspond roughly to four space differences as shown in Fig. 7.

Continuing,

$$\left[\frac{\overline{\underset{\sim}{R}}_\ell}{2\rho j}\frac{\partial p}{\partial k} + \frac{1}{2M}\frac{\partial}{\partial k}\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell \ {}_1q_A\right)\right]^n_{k,\ell-\frac{1}{2}} = \left[\frac{\overline{\underset{\sim}{R}}_\ell}{2\rho j}\right]^n_{k,\ell-\frac{1}{2}} \Delta p^{cn}_{k,\ell-\frac{1}{2}}$$

$$+ \frac{\Delta\left(\hat{R}\,\overline{\underset{\sim}{R}}_\ell \ {}_1q\right)^n_{k,\ell-\frac{1}{2}}}{M_{k+\frac{1}{2},\ell-\frac{1}{2}} + M_{k-\frac{1}{2},\ell-\frac{1}{2}}} ,$$

$$\left[\frac{\overline{\underset{\sim}{R}}_k}{2\rho j}\frac{\partial p}{\partial \ell} + \frac{1}{2M}\frac{\partial}{\partial \ell}\left(\hat{R}\,\overline{\underset{\sim}{R}}_k \ {}_2q_B\right)\right]^n_{k-\frac{1}{2},\ell} = \left[\frac{\overline{\underset{\sim}{R}}_k}{2\rho j}\right]^n_{k-\frac{1}{2},\ell} \delta p^{cn}_{k-\frac{1}{2},\ell}$$

$$+ \frac{\delta\left(\hat{R}\,\overline{\underset{\sim}{R}}_k \ {}_2q\right)^n_{k-\frac{1}{2},\ell}}{M_{k-\frac{1}{2},\ell+\frac{1}{2}} + M_{k-\frac{1}{2},\ell-\frac{1}{2}}} ,$$

with analogous expressions for the quantities containing ${}_3q_A$ and ${}_4q_B$.

$$\Delta\left(\hat{R}\,\overline{\mathbf{R}}_{\ell\,1}q\right)^n_{k,\ell-\frac{1}{2}} = \hat{R}_{k+\frac{1}{2},\ell-\frac{1}{2}}\,\delta\overline{\mathbf{R}}^{\,n}_{k+\frac{1}{2},\ell-\frac{1}{2}}\,{}_1q^n_{k+\frac{1}{2},\ell-\frac{1}{2}}$$

$$- \hat{R}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,\delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,{}_1q^n_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$$\delta\left(\hat{R}\,\overline{\mathbf{R}}_{k\,2}q\right)^n_{k-\frac{1}{2},\ell} = \hat{R}_{k-\frac{1}{2},\ell+\frac{1}{2}}\,\Delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell+\frac{1}{2}}\,{}_2q^n_{k-\frac{1}{2},\ell+\frac{1}{2}}$$

$$- \hat{R}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,\Delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,{}_2q^n_{k-\frac{1}{2},\ell-\frac{1}{2}}.$$

The differencing of the next two expressions is not as straightforward as was the case for the previous two.

$$\left[\frac{\mathbf{R}_\ell}{2\rho j}\right]^n_{k,\ell-\frac{1}{2}} = \frac{\delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell-\frac{1}{2}} + \delta\overline{\mathbf{R}}^{\,n}_{k+\frac{1}{2},\ell-\frac{1}{2}}}{2\left[\dfrac{j^n_{k-\frac{1}{2},\ell-\frac{1}{2}}}{\tau^n_{k-\frac{1}{2},\tau-\frac{1}{2}}} + \dfrac{j^n_{k+\frac{1}{2},\ell-\frac{1}{2}}}{\tau^n_{k+\frac{1}{2},\ell-\frac{1}{2}}}\right]},$$

$$\left[\frac{\mathbf{R}_\ell}{2\rho j}\right]^n_{k-\frac{1}{2},\ell} = \frac{\Delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell-\frac{1}{2}} + \Delta\overline{\mathbf{R}}^{\,n}_{k-\frac{1}{2},\ell+\frac{1}{2}}}{2\left[\dfrac{j^n_{k-\frac{1}{2},\ell-\frac{1}{2}}}{\tau^n_{k-\frac{1}{2},\ell-\frac{1}{2}}} + \dfrac{j^n_{k-\frac{1}{2},\ell+\frac{1}{2}}}{\tau^n_{k-\frac{1}{2},\ell+\frac{1}{2}}}\right]}.$$

The differencing used to represent $\overline{\mathbf{R}}_\ell/2\rho j$ and $\overline{\mathbf{R}}_k/2\rho j$ in the preceding equations is a departure from previous practice. Increased importance has been given to j, the area Jacobian. More specifically, the relationship between j and the derivatives $\mathbf{R}_k$, $\mathbf{R}_\ell$ is weighted in favor of j. j is the Jacobian of the transformation from Eulerian to Lagrangian coordinates and is most simply defined at the center of a zone. $\mathbf{R}_k$ and $\mathbf{R}_\ell$, to be consistent with this j, should also be defined at the center of a zone. Thus, if in a

given difference equation both $\mathbf{R}_\ell$ (or $\overline{\mathbf{R}}_\ell$) and j are desired along the side of a zone and j is calculated by averaging the central j's on either side, then $\mathbf{R}_\ell$ should also be calculated by averaging the central $\mathbf{R}_\ell$'s on either side. So where previous codes used

$$\left[\overline{\mathbf{R}}_\ell\right]^n_{k,\ell-\frac{1}{2}} = \delta\overline{\mathbf{R}}^{\,n}_{k,\ell-\frac{1}{2}}$$

we use instead

$$\left[\overline{\mathbf{R}}_\ell\right]^n_{k,\ell-\frac{1}{2}} = \frac{1}{4}\left[\delta\overline{\mathbf{R}}^{\,n}_{k-1,\ell-\frac{1}{2}} + 2\delta\overline{\mathbf{R}}^{\,n}_{k,\ell-\frac{1}{2}} + \delta\overline{\mathbf{R}}^{\,n}_{k+1,\ell-\frac{1}{2}}\right]$$

as is demonstrated by the differencing of the momentum equation. This insures, for example, that the momentum equation behaves reasonably for the case of Fig. 8a.

In earlier codes the interface between the two zones would move toward the high pressure zone, instead of away as will occur with the present system.

Using $\mathbf{u}^{n+\frac{1}{2}}_{k,\ell}$ from (6.3), $\mathbf{R}^{n+1}_{k,\ell}$ follows from

$$\mathbf{R}^{n+1}_{k,\ell} = \mathbf{R}^n_{k,\ell} + Dt^{n+\frac{1}{2}}\,\mathbf{u}^{n+\frac{1}{2}}_{k,\ell}. \tag{6.4}$$

This completes the differencing of the momentum equation.

We now proceed to the calculation of the remaining quantities in T storage.

$$j^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \Delta R^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,\delta Z^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} - \delta R^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,\Delta Z^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}. \tag{6.5}$$

For the volume of a zone one uses a radian slice in cylindrical coordinates and a unit thick slice in Cartesian coordinates.

$$J^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \hat{R}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\,j^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}.$$

Since

$$R_{k-\frac{1}{2},\ell-\frac{1}{2}} = \tfrac{1}{4}\left[R_{k,\ell} + R_{k-1,\ell} + R_{k,\ell-1} + R_{k-1,\ell-1}\right],$$

this is not the true volume of a zone for the cylindrical case.

$$J_{true} = \frac{1}{6}\Delta\underset{\sim}{R}_{k-\frac{1}{2},\ell} \cdot \delta\underset{\sim}{\bar{R}}_{k,\ell-\frac{1}{2}}\left(\hat{R}_{k-1,\ell} + \hat{R}_{k,\ell} + \hat{R}_{k,\ell-1}\right)$$

$$+ \frac{1}{6}\Delta\underset{\sim}{R}_{k-\frac{1}{2},\ell-1} \cdot \delta\underset{\sim}{\bar{R}}_{k-1,\ell-\frac{1}{2}}\left(\hat{R}_{k-1,\ell} + \hat{R}_{k-1,\ell-1} + \hat{R}_{k,\ell-1}\right).$$

J_{true} and j do not necessarily approach zero together. This is particularly true along the axis of a cylinder where a pathological zone could easily have a positive volume together with a negative area, as in Fig. 8b. Since we are trying to write difference equations which are reasonably valid even when negative volumes are present, we cannot allow this nonsense situation.

The expression used for J is in agreement with our general policy of giving increased importance to j, the Jacobian of the transformation from Eulerian to Lagrangian space. In the momentum equation $\Delta\underset{\sim}{R}$ and $\delta\underset{\sim}{R}$ were averaged in a special way depending on how j was averaged. Here, in analogous fashion, J is calculated to favor j.

The specific volume follows directly.

$$\tau^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = J^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} \Big/ M_{k-\frac{1}{2},\ell-\frac{1}{2}}. \tag{6.6}$$

The next step is to advance the four q's to time n + 1. Using the relations (4.1) and (4.2) as a base we will define as follows:

$$(\Delta u_-^A)_{k-\frac{1}{2},\ell-\frac{1}{2}} = \min\left\{\frac{\delta\underset{\sim}{\bar{R}}^n_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \Delta\underset{\sim}{u}^{n-\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}{\left|\delta\underset{\sim}{\bar{R}}^n_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$$(\delta u_-^B)_{k-\frac{1}{2},\ell-\frac{1}{2}} = \min\left\{\frac{-\Delta\underset{\sim}{\bar{R}}^n_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \delta\underset{\sim}{u}^{n-\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}{\left|\Delta\underset{\sim}{\bar{R}}^n_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$$\left|\Delta\Delta u_-^A\right|_{k,\ell-\frac{1}{2}} = \left|(\Delta u_-^A)_{k+\frac{1}{2},\ell-\frac{1}{2}} - (\Delta u_-^A)_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|,$$

$$\left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell} = \left|(\delta u_-^B)_{k-\frac{1}{2},\ell+\frac{1}{2}} - (\delta u_-^B)_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|,$$

$$\left|\Delta\Delta u_-^A\right|_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{1}{2}\left\{\left|\Delta\Delta u_-^A\right|_{k,\ell-\frac{1}{2}} + \left|\Delta\Delta u_-^A\right|_{k-1,\ell-\frac{1}{2}}\right\},$$

$$\left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{1}{2}\left\{\left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell} + \left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell-1}\right\},$$

$$(\Delta u_-^A)_1 = \min\left\{\frac{\delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \Delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell}}{\left|\delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$$(\delta u_-^B)_2 = \min\left\{\frac{-\Delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k,\ell-\frac{1}{2}}}{\left|\Delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$$(\Delta u_-^A)_3 = \min\left\{\frac{\delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \Delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-1}}{\left|\delta\underset{\sim}{\bar{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$$\left(\delta u_-^B\right)_4 = \min\left\{\frac{-\Delta\bar{\underset{\sim}{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \delta \underset{\sim}{u}^{n+\frac{1}{2}}_{k-1,\ell-\frac{1}{2}}}{\left|\Delta\bar{\underset{\sim}{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right|}, 0\right\},$$

$${}_1q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{-2}{\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}\left(\Delta u_-^A\right)_1 \cdot \left|\Delta\Delta u_-^A\right|_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$${}_2q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{-2}{\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}\left(\delta u_-^B\right)_2 \cdot \left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$${}_3q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{-2}{\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}\left(\Delta u_-^A\right)_3 \cdot \left|\Delta\Delta u_-^A\right|_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$${}_4q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{-2}{\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}\left(\delta u_-^B\right)_4 \cdot \left|\delta\delta u_-^B\right|_{k-\frac{1}{2},\ell-\frac{1}{2}}. \tag{6.7}$$

Note that in producing these q's at time n + 1, we use $\underset{\sim}{R}^n$, $\underset{\sim}{R}^{n+\frac{1}{2}}$, $\underset{\sim}{u}^{n-\frac{1}{2}}$, $\underset{\sim}{u}^{n+\frac{1}{2}}$, and $\tau^{n+\frac{1}{2}}$. It doesn't look like a very accurate procedure. Indeed it is not if one considers that we are just differencing the expressions (4.1) and (4.2). The point to remember is that as far as the conservation equations are concerned, all we need is a viscosity which vanishes sufficiently rapidly in front and in back of a shock. The defined expressions have

this property. We can consider that these q's are an accurate representation of some viscosity, slightly different from q_A and q_B, which we have not bothered to find differential expressions for. Part of the reason for using $\underset{\sim}{R}$, $\underset{\sim}{u}$, and τ at these various times is simply a matter of convenience related to details of programming. This tensor viscosity was introduced into an existing code. However, one could never produce q_A and q_B at n + 1 since the latest velocity available is $\underset{\sim}{u}^{n+\frac{1}{2}}$.[11]

Previous codes calculated q's at $n + \frac{1}{2}$. As far as the momentum equation is concerned, it doesn't matter what time label you put on q, you use the same quantity. In the energy equation, however, it does make a difference, as we see in the following.

$$(\delta\bar{\underset{\sim}{R}} \cdot \Delta\underset{\sim}{u})_-^{n+\frac{1}{2}} = \min\left\{\delta\bar{\underset{\sim}{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \Delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}, 0\right\},$$

$$(-\Delta\bar{\underset{\sim}{R}} \cdot \delta\underset{\sim}{u})_-^{n+\frac{1}{2}} = \min\left\{-\Delta\bar{\underset{\sim}{R}}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \cdot \delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}, 0\right\},$$

$$q_A^{n+\frac{1}{2}} = \frac{1}{4}\left({}_1q^n_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_1q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_3q^n_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_3q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right),$$

$$q_B^{n+\frac{1}{2}} = \frac{1}{4}\left({}_2q^n_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_2q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_4q^n_{k-\frac{1}{2},\ell-\frac{1}{2}} + {}_4q^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right),$$

$$D\epsilon(q)^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \frac{-\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\, Dt^{n+\frac{1}{2}}}{j^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}}\left[q_A^{n+\frac{1}{2}}(\delta\bar{\underset{\sim}{R}} \cdot \Delta\underset{\sim}{u})_-^{n+\frac{1}{2}} + q_B^{n+\frac{1}{2}}\,(-\Delta\bar{\underset{\sim}{R}} \cdot \delta\underset{\sim}{u})_-^{n+\frac{1}{2}}\right],$$

$$\tilde{\epsilon}^{n+\frac{1}{2}} = \epsilon^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} + \frac{1}{2}\left[-p^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} D\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} + D\epsilon(q)^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} + D\epsilon(s)^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right].$$

$$\tilde{p}^{n+\frac{1}{2}} = p\left(\tilde{\epsilon}^{n+\frac{1}{2}}, \tau^{n+\frac{1}{2}}\right),$$

$$\epsilon^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}} = \epsilon^{n}_{k-\frac{1}{2},\ell-\frac{1}{2}} - \tilde{p}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} D\tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} + D\epsilon(q)^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} + D\epsilon(s)^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}. \qquad (6.8)$$

Defining the q's at $n+1$ makes it clear that in the energy equation one should average backward in time so as to produce q's at $n+\frac{1}{2}$. This system is second-order accurate in time with respect to the artificial viscosity as well as the pressure. Previous explicit codes were only first order in the viscosity.

We have now advanced all the quantities in T storage from time n to $n+1$ and have thus advanced the system one time step. By iteration of this procedure the system is moved through time as far as is desired, in theory anyway.

VII. Δt CONTROL

A least upper bound on the magnitude of Δt^n is provided by the stability requirements of the sound equation (5.5) and the q stability equation (5.6). These equations will be differenced in a straightforward explicit manner that

corresponds reasonably closely to the way **the hydrodynamic equations are** differenced. **The differential equations under consideration are themselves** only approximations of the hydrodynamic **differential equations. Thus straight-**forward differencing should provide results within **the order of approximation** of the differential equations.

The stability analysis will be done in matrix formulation. **A short** résumé of this procedure is as follows. Let

$$F^{(n)} \sim \left(\cdots f^{n}_{k,\ell-1}, f^{n}_{k,\ell}, f^{n}_{k,\ell+1}, \cdots\right).$$

Then the differenced equation is written in the form

$$F^{(n+1)} = UF^{(n)} \text{ and } \therefore F^{(n)} = U^n F^{(0)}$$

where U is a matrix. Thus repeated applications of U advance the system through time. This will also be true of any errors that are introduced, i. e., let

$$F^{(n)} \to F^{(n)} + F_e^{(0)},$$

then

$$F_e^{(n)} = U^n F_e^{(0)}.$$

Expand $F_e^{(0)}$ in terms of the eigenfunctions of U,

$$UF(m) = U'_m F(m),$$

$$F_e^{(0)} = \sum_m a_m F(m).$$

Then one obtains

$$F_e^{(n)} = \sum_m a_m \left(U'_m\right)^n F(m)$$

and we see that unless

$$|U'_m| \leq 1$$

the mth component of the error will grow exponentially. Thus stability requires us to use matrices U whose eigenvalues satisfy the above condition.

Consider an operator of the form

$$\mathcal{H} = a_1 \frac{\partial^2}{\partial k^2} + 4a_2 \frac{\partial^2}{\partial k \partial \ell} + a_3 \frac{\partial^2}{\partial \ell^2}$$

where a_1, a_2, and a_3 are constants, a_1 and $a_3 > 0$. Let f be an arbitrary function, then

$$[\mathcal{H}f]_{k,\ell} = a_1[f_{k+1,\ell} - 2f_{k,\ell} + f_{k-1,\ell}]$$
$$+ a_2[f_{k+1,\ell+1} + f_{k-1,\ell-1} - f_{k-1,\ell+1} - f_{k+1,\ell-1}]$$
$$+ a_3[f_{k,\ell+1} - 2f_{k,\ell} + f_{k,\ell-1}].$$

The matrix H will be now defined by setting

$$(HF)_{k,\ell} = [\mathcal{H}f]_{k,\ell}.$$

The eigenfunctions of H are just

$$F(m_1m_2) = (\ldots f_{k,\ell}(m_1m_2) \ldots),$$

$$f_{k,\ell}(m_1m_2) = f_0 \exp(im_1k + im_2\ell).$$

Apply H to $F(m_1m_2)$ to find the eigenvalues, H', of H.

$$HF(m_1m_2) = H'(m_1m_2)\, F(m_1m_2),$$

$$\left(HF(m_1m_2)\right)_{k,\ell} = a_1f_{k,\ell}(m_1m_2)\left[e^{im_1} - 2 + e^{-im_1}\right]$$
$$+ a_2f_{k,\ell}(m_1m_2)\left[\exp(im_1 + im_2) + \exp(im_1 - im_2)\right.$$
$$\left. - \exp(-im_1 + im_2) - \exp(im_1 - im_2)\right]$$
$$+ a_3f_{k,\ell}(m_1m_2)\left[e^{im_2} - 2 + e^{-im_2}\right].$$

The first bracket can be written as

$$4i^2 \frac{\left(\exp(i\frac{m_1}{2}) - \exp(-i\frac{m_1}{2})\right)}{2i} \frac{\left(\exp(i\frac{m_1}{2}) - \exp(-i\frac{m_1}{2})\right)}{2i} = -4 \sin^2 \frac{m_1}{2}.$$

Similarly the third bracket is just

$$-4 \sin^2 \frac{m_2}{2}.$$

The second bracket can be written as

$$4i^2 \frac{\left(e^{im_1} - e^{-im_1}\right)}{2i} \frac{\left(e^{im_2} - e^{-im_2}\right)}{2i} = -4 \sin m_1 \sin m_2.$$

Thus the eigenvalues of H are given by

$$H'(m_1m_2) = -4\left[a_1 \sin^2\frac{m_1}{2} + a_3 \sin^2\frac{m_2}{2} + a_2 \sin m_1 \sin m_2\right].$$

An upper limit to the range of values that the magnitude of m_1 or m_2 may have results from k and ℓ being discrete variables. A sawtooth behavior describes the shortest wavelength that a finite grid may describe. This corresponds to

$$m_1,\ m_2 = \pm\pi$$

and is the dominant mode occurring in most instabilities. A lower limit on the magnitude of m, n is provided by the fact that the differential equations we are applying this analysis to were derived on the basis of $m \cdot n$ being significantly larger than m or n individually. This does not leave much room since π is not a very large number. It would seem that at most the analysis could apply only to the modes.

$$(m_1m_2) = (\pm\pi, \pm\pi), \left(\pm\pi, \pm\frac{\pi}{2}\right), \left(\pm\frac{\pi}{2}, \pm\pi\right)$$

and probably only to the first or dominant mode.[12] For all three of these modes, however, the cross difference term vanishes. Thus it will be dropped, allowing us to consider that H' has the properties

$$H' < 0, \; H'_{min} = -4(a_1 + a_3).$$

It is this vanishing of the cross-difference term that insures stability of the difference equations for the scalar q system. In the differential equation case it was the nondefinite contribution of the cross-derivative term that allowed the differential equations to become unstable in certain situations.

We wish to find the stability criteria for explicit differencing of the following two equations.

$$\text{Case I}: \frac{\partial^2 f}{\partial t^2} = f,$$

$$\text{Case II}: \frac{\partial f}{\partial t} = f.$$

The first corresponds to the sound equation and the second to the q stability equation.

Case I

Performing the k, ℓ differencing gives us

$$\frac{\partial^2 F}{\partial t^2} = HF.$$

Next, differencing over time,

$$\frac{F^{n+1} - 2F^n + F^{n-1}}{Dt^2} = HF^n$$

or

$$[U - 2 + U^{-1} - Dt^2 H]\, F^n = 0,$$

$$\tfrac{1}{2}U^2 - \left[\frac{Dt^2}{2} H + 1\right] U + \tfrac{1}{2} = 0,$$

$$a = c = \tfrac{1}{2}, \; b = -\frac{Dt^2}{2} H - 1, \; U = -b \pm \sqrt{b^2 - 1}.$$

From here on we consider the operators to be replaced by their eigenvalues, designated by a prime on the quantity.

For $b'^2 > 1$, $|U'|$ can obviously be > 1 **which is unsatisfactory. If we** require

$$-1 \le b' \le 1,$$

then

$$U' = -b' \pm i\sqrt{1 - b'^2},$$

$$|U'|^2 = b'^2 + 1 - b'^2 = 1,$$

showing the differencing is stable if this condition is satisfied.

$$-1 \le -\frac{Dt^2}{2} H' - 1 \le 1.$$

Since H' is negative, the lower limit is satisfied. **The upper limit provides** the condition

$$Dt^2 \le -4/H'.$$

Thus for explicit differencing of

$$\frac{\partial^2 f}{\partial t^2} = a_1 \frac{\partial^2 f}{\partial k^2} + 4a_2 \frac{\partial^2 f}{\partial k \partial \ell} + a_3 \frac{\partial^2 f}{\partial \ell^2}$$

we obtain the general condition (if $H' \le 0$):

$$Dt^2 \le \left[a_1 \sin^2 \frac{m_1}{2} + a_2 \sin m_1 \sin m_2 + a_3 \sin^2 \frac{m_2}{2}\right]^{-1}$$

which in our case reduces to

$$Dt^2 \le [a_1 + a_3]^{-1}. \quad \text{(Case I.)} \tag{7.1}$$

Case II

$$\frac{F^{n+1} - F^n}{Dt} = HF^n, \; [U - 1 - DtH]\, F^n = 0,$$

$$U = 1 + DtH.$$

Stability requires

$$-1 \le U' \le 1.$$

Assume $H' < 0$, then the upper limit is satisfied. The lower limit gives us

$$-DtH' \leq 2.$$

Thus for explicit differencing of

$$\frac{\partial f}{\partial t} = a_1 \frac{\partial^2 f}{\partial k^2} + 4a_2 \frac{\partial^2 f}{\partial k \partial \ell} + a_3 \frac{\partial^2 f}{\partial \ell}$$

we obtain the general condition (if $H' < 0$):

$$Dt \leq \left[2\left(a_1 \sin^2 \frac{m_1}{2} + a_2 \sin m_1 \sin m_2 + a_3 \sin^2 \frac{m_2}{2}\right)\right]^{-1}$$

which in our case reduces to

$$Dt \leq [2(a_1 + a_3)]^{-1}. \qquad \text{(Case II.)} \tag{7.2}$$

The sound equation (5.5) can be written as

$$\frac{\partial^2 f}{\partial t^2} = \frac{c^2}{j^2}\left[\delta \underset{\sim}{R}^2 \frac{\partial^2 f}{\partial k^2} - 2\delta \underset{\sim}{R} \cdot \Delta \underset{\sim}{R} \frac{\partial^2 f}{\partial k \partial \ell} + \Delta \underset{\sim}{R}^2 \frac{\partial^2 f}{\partial \ell^2}\right].$$

Thus

$$a_1 = \frac{c^2}{j^2} \delta \underset{\sim}{R}^2, \quad a_3 = \frac{c^2}{j^2} \Delta \underset{\sim}{R}^2,$$

giving the following for the Courant stability condition (Case I).

$$Dt_c^2 \leq \frac{j^2}{c^2\left(\delta \underset{\sim}{R}^2 + \Delta \underset{\sim}{R}^2\right)}.$$

For a γ-law gas,

$$p = A(S)\rho^{\gamma},$$

so that

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = \gamma p \tau \quad (\gamma\text{-law}).$$

The equations of state used are not simple γ-law gases in general. However, to obtain a simple expression for c^2 one can calculate an effective γ on the assumption that one does have a γ-law gas and then use the preceding equation; that is, solve

$$p = (\gamma_{eff} - 1)\epsilon/\tau$$

for γ_{eff},

$$\gamma_{eff} = \left(1 + \frac{p\tau}{\epsilon}\right).$$

Thus a Courant Dt_c is calculated for every zone of the problem.

$$c^2 = p^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} + \frac{\left(p^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}} \tau^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right)^2}{\epsilon^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-\frac{1}{2}}},$$

$$Dt_c^2 = K \frac{\left(j^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right)^2}{c^2\left[\left(\delta \underset{\sim}{R}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right)^2 + \left(\Delta \underset{\sim}{R}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}\right)^2\right]}, \tag{7.3}$$

where K, the Courant number squared, provides a fudge factor sufficient for any possibility. A satisfactory value for K seems to be

$$K = 0.25,$$

corresponding to a Courant number of 0.5. The minimum of these Dt_c^2 is saved and square-rooted producing a single Dt_c^{min}.

From the q stability equation (5.6) we see that

$$a_1 = -\frac{\tau}{j} A \underset{\sim}{R}_{\ell}^2, \quad a_3 = -\frac{\tau}{j} B \underset{\sim}{R}_k^2.$$

Thus

$$Dt_q \leq \frac{-\rho j}{2\left(A \underset{\sim}{R}_{\ell}^2 + B \underset{\sim}{R}_k^2\right)}$$

or

$$Dt_q = \frac{-j^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}}}{10[(\delta\bar{\underset{\sim}{R}} \cdot \Delta\underset{\sim}{u})_- + (-\Delta\bar{\underset{\sim}{R}} \cdot \delta\underset{\sim}{u})_-]} \tag{7.4}$$

where

$$(\delta\bar{\underset{\sim}{R}} \cdot \Delta\underset{\sim}{u})_- = \min\left[\Delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell} \cdot \delta\bar{\underset{\sim}{R}}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}},\ \Delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-\frac{1}{2},\ell-1} \cdot \delta\bar{\underset{\sim}{R}}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}},\ 0\right],$$

$$(-\Delta\bar{\underset{\sim}{R}} \cdot \delta\underset{\sim}{u})_- = \min\left[-\delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k,\ell-\frac{1}{2}} \cdot \Delta\bar{\underset{\sim}{R}}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}},\ -\delta\underset{\sim}{u}^{n+\frac{1}{2}}_{k-1,\ell-\frac{1}{2}} ; \Delta\bar{\underset{\sim}{R}}^{n+1}_{k-\frac{1}{2},\ell-\frac{1}{2}},\ 0\right],$$

and where the 10 corresponds to

$$10 = 4c_0^2 + 2.$$

The extra 2 added to the value of $4c_0^2$ insures that Dt_q is always below its upper limit.

Consider the expression for Dt_q. Let the Dt of the problem be equal to the Dt_q of some zone and let that zone be collapsing in both directions, then

$$\frac{Dj}{Dt_q} = (\delta\bar{\underset{\sim}{R}} \cdot \Delta\underset{\sim}{u})_- + (-\Delta\bar{\underset{\sim}{R}} \cdot \delta\underset{\sim}{u})_- .$$

Substituting in (7.4) there results

$$-Dj = j/10.$$

We see that this condition limits the amount of area any zone can lose on any one time step to approximately 10%. (It is approximate since this Dt_q is used on the next time cycle.) Intuitively this seems like a reasonable accuracy condition as well as a reasonable stability criterion no matter what viscosity we are using.

The minimum of these Dt_q is saved and one then calculates

$$Dt^{n+3/2} = \min\left[Dt_q^{min},\ Dt_c^{min}\right].$$

$Dt^{n+3/2}$ is used for $Dt^{n+\frac{1}{2}}$ on the next time cycle, **thus insuring the stability** of the difference equations. These stability requirements also seem to provide a sufficient condition for the accuracy of integration.

VIII. BOUNDARY TREATMENT

The special routines needed to calculate the momentum equation along the boundary of the system are held to a minimum by extending the medium a half zone outside and defining external pressures at this point. Then one can use the general momentum equation (6.3) for boundary points as well as interior points. This has the disadvantage that when people make up a problem they have to remember that the code assumes there is material outside of the grid laid out on the system. Sometimes they don't remember, but if the zoning is fine enough to decently integrate the system, it usually won't cause a significant error.

By calculating the work done on the boundary of the system an energy check may be put into the code. From equation (3.5) one can deduce a suitable expression for this work.

$$DW^{n-\frac{1}{2}}_{k,\ell-\frac{1}{2}} = \pm Dt^{n-\frac{1}{2}}[(p + q_A)\underset{\sim}{u} \cdot \hat{R}\ \delta\bar{\underset{\sim}{R}}]^{n-\frac{1}{2}}_{k,\ell-\frac{1}{2}}.$$

$$DW^{n-\frac{1}{2}}_{k-\frac{1}{2},\ell} = \pm Dt^{n-\frac{1}{2}}[(p + q_B)\underset{\sim}{u} \cdot \hat{R}\ \Delta\bar{\underset{\sim}{R}}]^{n-\frac{1}{2}}_{k-\frac{1}{2},\ell}.$$

By summing $DW^{n-\frac{1}{2}}$ along the boundary and adding to W^{n-1}, one obtains W^n, the amount of work done on the system since the problem started.

The total internal and kinetic energy can be calculated from

$$(I.\,E.)^n = \sum_{k,\ell} M_{k-\frac{1}{2},\ell-\frac{1}{2}} \epsilon^n_{k-\frac{1}{2},\ell-\frac{1}{2}},$$

$$(K.\,E.)^{n+\frac{1}{2}} = \sum_{k,\ell} \frac{1}{8} M_{k-\frac{1}{2},\ell-\frac{1}{2}} \left[\left(\underset{\sim}{u}^{n+\frac{1}{2}}_{k,\ell}\right)^2 + \left(\underset{\sim}{u}^{n+\frac{1}{2}}_{k-1,\ell}\right)^2 + \left(\underset{\sim}{u}^{n+\frac{1}{2}}_{k,\ell-1}\right)^2 + \left(\underset{\sim}{u}^{n+\frac{1}{2}}_{k-1,\ell-1}\right)^2\right].$$

The kinetic energy is desired at time n.

$$(K.\,E.)^n = \frac{(K.\,E.)^{n-\frac{1}{2}} Dt^{n+\frac{1}{2}} + (K.\,E.)^{n+\frac{1}{2}} Dt^{n-\frac{1}{2}}}{Dt^{n-\frac{1}{2}} + Dt^{n+\frac{1}{2}}},$$

then define

$$C_E^n = (I.\,E.)^n + (K.\,E.)^n - W^n,$$

$$DC_E^{n-\frac{1}{2}} = C_E^n - C_E^{n-1}.$$

C_E^n is the estimate at time n of the total energy in the system at time n = 0. If the integration of the system were perfect, it would be a constant. $DC_E^{n-\frac{1}{2}}$ is the gain of energy per time cycle due to the imperfect integration of the difference equations. A large fluctuation in $DC_E^{n-\frac{1}{2}}$ is an indication of a possible machine (or other) error.

IX. CONCLUSION

Writing difference equations is an art. There is only a certain amount of rough analysis that one can do and only a limited amount of foresight that one can provide. The procedure is to write a reasonable set of difference equations, code them up for a machine, and run some problems. When the code breaks down, you examine what went wrong and then fix up your difference equations and code so that that particular trouble won't happen again. Next you run more problems until the code breaks down again. Then you fix this trouble up, etc., etc.

The difference equations in this report are not just a "reasonable set of difference equations." They are the result of running production problems on the 701, 704, 709, 7090, 7094, LARC, and STRETCH. They comprise a stable, well-behaved system. Their bias is that they are built about a certain class of problems. Nevertheless, this class of problems was quite large and if Lagrangian hydrodynamics is suitable for a given problem, these equations ought to produce satisfactory results. At the very least, they should provide a better than "reasonable set of difference equations" to start one's own code development with.

X. APPENDIX — SAMPLE PROBLEM

A small problem was run to demonstrate the behavior of the difference equations in this report. Initially the system is in the form of an oblate spheroid (doorknob shape) of gas at temperature absolute zero. At time $t = 0$ a pressure profile (Fig. 9) is uniformly applied to the outside of the spheroid. (The units of measurement used in this example are completely arbitrary. Choose any self-consistent set desired.) The future behavior of the system is shown qualitatively by the following series of plots of the Lagrangian grid laid out on the system. These plots were made on a cathode ray tube by a service code which processes the data produced by the main code. Detailed quantitative information would consist of pages of numbers, the values of the quantities in T storage, given at various times. The problem isn't of sufficient interest to merit that treatment and besides, the zoning used was relatively coarse. This kept the running time of the problem to a minimum. If one were really interested in the details of the behavior of this system, one would have to run a second more finely zoned problem. Relative to the scale of your detailed interest and the general credibility of the numerical model, they must give the same answer for one to believe the results.

The Lagrangian grid is laid out on one quarter of a cross section through the axis of the system. The Z axis is made a reflecting line and the R axis a reflecting plane. From symmetry this will then describe the behavior of the entire system.

Lagrangian codes are best for "smooth" hydrodynamic flow such as the flow occurring during the early stages of the problem while the spheroid is being compressed. More generally, the concept of "smooth" flow may be related to the constraints put on the system by confining little volumes of gas

into quadrilateral shapes. In the presence of turbulence, for example, a volume of gas which is initially four-sided may in reality change into a spiraling threadlike structure. A Lagrangian code demands that this structure be represented by a four-sided figure. Thus Lagrangian codes are not suited for a "nonsmooth" flow of this type. The problem of how to damp out motion smaller than the scale of the finite grid still remains with you even if you use Eulerian difference equations, although there the difficulty is not so evident. Artificial viscosities were designed to convert kinetic energy into internal energy for the case of shocks. Are they correctly transforming the kinetic energy of small scale turbulence into internal energy ?

By time $t = 10$ the Lagrangian grid around the center of the spheroid has started to "scramble" a bit. One would be skeptical of detailed information given by the code in this region. Even though as time increases this area seems to get progressively worse, the difficulty is really confined to a relatively small area of the problem and there is no reason to doubt the correctness of the general behavior of the system. This points up the advantage of having difference equations stable enough to run even though the Lagrangian grid does get irregular in some areas.

The external pressure profile applied to this system corresponds to the spheroid being pushed by a gas of zero density. Thus the surface of the system is Taylor unstable. The final state of the system consists of one large jet squirting out the axis of the system. More correctly, there are two jets, one on each side of the reflecting plane if one considers the entire system.

Lagrangian hydrodynamics possesses two major advantages over Eulerian hydrodynamics. First, the zoning stays with the material. If you compress a system to smaller size, the system retains the same amount of

definition. In Eulerian hydrodynamics the entire system could end up in one zone. Second, the interfaces between different materials are clearly defined. In Eulerian hydrodynamics interfaces tend to diffuse and one must do special things to take care of this. While there is only one material present in the sample problem the surrounding void can be considered as a second material, as was done previously in the Taylor instability discussion. The boundary of the system can thus be considered as a clearly defined material interface.

REFERENCES

*This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] These are people who are or have been at Livermore and whose work is thus easily available to the author.

[2] A complete discussion of the tensor artificial viscosity will be found in the next section (Section IV).

[3] Remember we are integrating over a radian slice in cylindrical coordinates. If the total volume of the system were integrated over, then a scalar q would also only appear in surface integrals.

[4] J. von Neumann and R. D. Richtmyer, J. Appl. Phys. 21, 232 (1950), or see the last chapter of Richtmyer's book Difference Methods For Initial Value Problems (Interscience, New York, 1957).

[5] The question as to whether the second coefficient of viscosity vanishes or not is apparently still open. That is, perhaps in a uniformly contracting system there is an entropy change. This, however, is a separate question. The artificial viscosity attempts to take care of shock heating only.

[6] For nonuniform zones this q will not vanish as desired. In order to take this into account the following q was tried originally.

$$\left(\frac{\partial u}{\partial R}\right)_- = \min\left[\frac{\partial u}{\partial R}\,,\ 0\right],$$

$$q \text{ tentative} = -c_0^2\rho\left(\frac{\partial u}{\partial R}\right)_- \left|\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial R}\right)_-\right|\left(\frac{\partial R}{\partial k}\right)^2 .$$

This q was unsatisfactory. Shocks were propagated through a large zone surrounded by small zones in an unphysical manner.

[7] Up to this point, the justification for calling this viscosity a "tensor" follows from its directional properties. Its relation to a formal tensor in Eulerian space is shown at the end of this section.

[8] Only if $\tau_t \geq 0$; if $\tau_t < 0$ lower order terms cause instabilities.

[9] In this process α is assumed to have twice the weight of an m_1 or m_2. If the final equation gives α as a quadratic function of m_1 and m_2, then this assumption is self-consistent and justified.

[10] A subscript of 1, 2, 3, or 4 on a variable will always refer to a location inside the quadrilateral as shown in the diagram.

[11] One change that might make an improvement in the system would be to calculate $|\Delta\Delta u_-^B|$ and $|\delta\delta u_-^B|$ at 1, 3 and 2, 4 respectively instead of using a common zone-centered quantity in each case.

[12] Just remember the whole approach is a "heuristic" one, i. e., the conditions derived are sufficient to insure stability in practice.

/hm

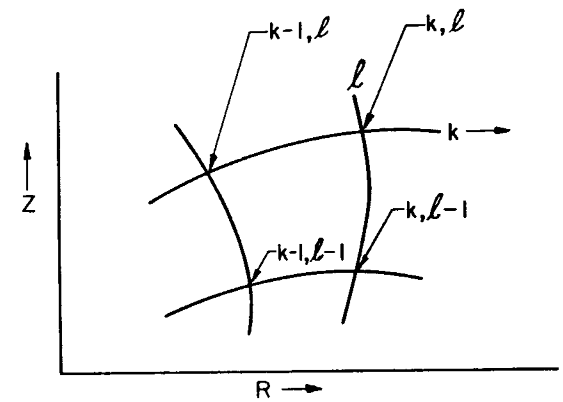

Fig. 1. Lagrangian coordinate network for differencing.

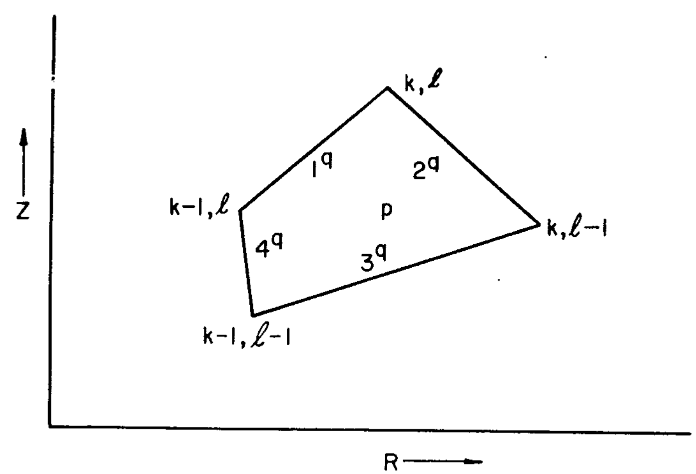

Fig. 2. Typical quadrilateral zone in discrete k, ℓ space.

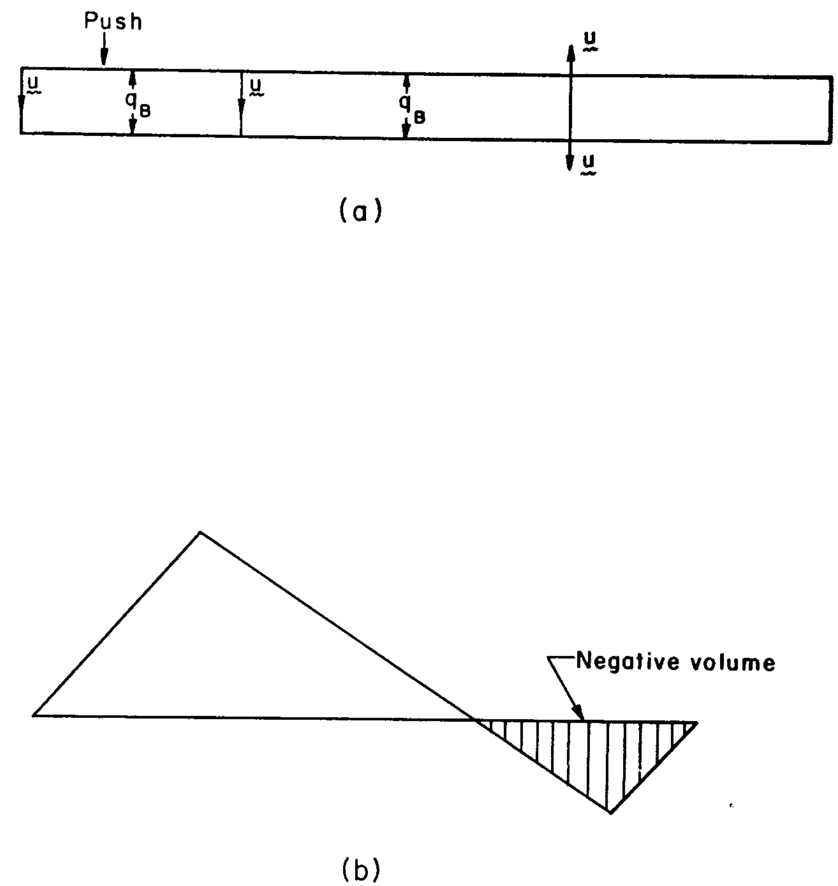

Fig. 3. (a) Long thin zone being collapsed on one side. (b) Irregular or "pathological" zone.

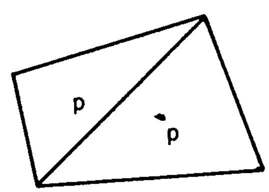

Fig. 4. Quadrilateral system reduced to triangular.

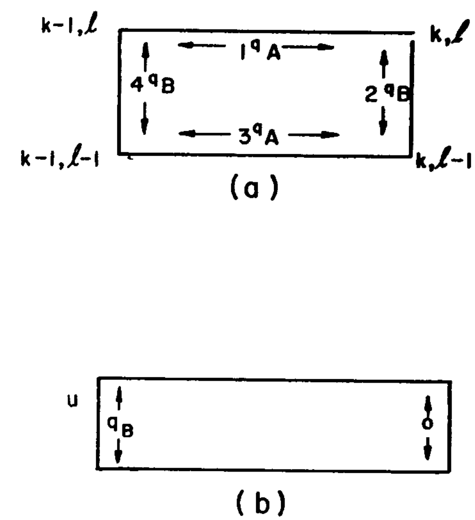

Fig. 5. (a) Quadrilateral zone used for difference equations in this paper. (b) Calculation of q's when a regular zone collapses on one side.

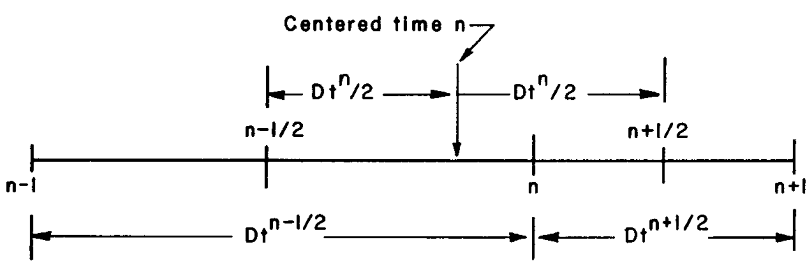

Fig. 6. Illustration of "centered time n."

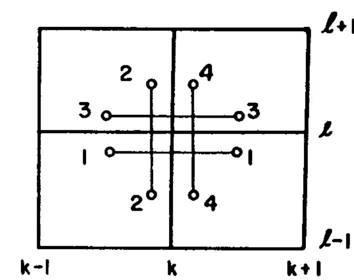

Fig. 7. Space differences corresponding to the four square-bracketed expressions in Eq. (6.3).

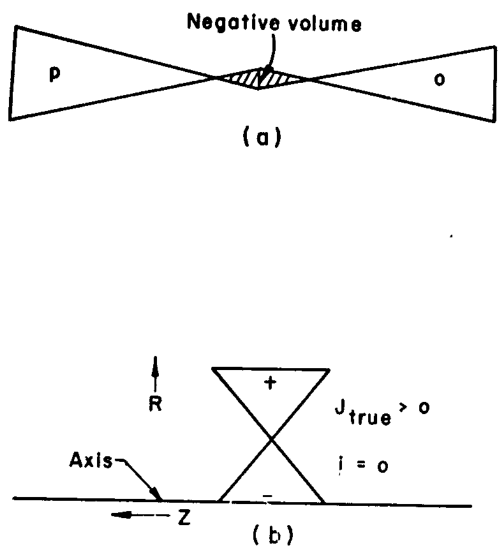

Fig. 8. (a) Pathological zone of a type for which momentum equation behaves reasonably. (b) Pathological zone having positive volume and zero area, a nonsense situation.

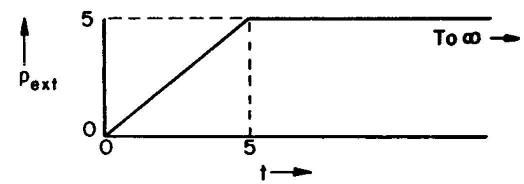

Fig. 9. Pressure applied externally to oblate spheroid of gas in example.

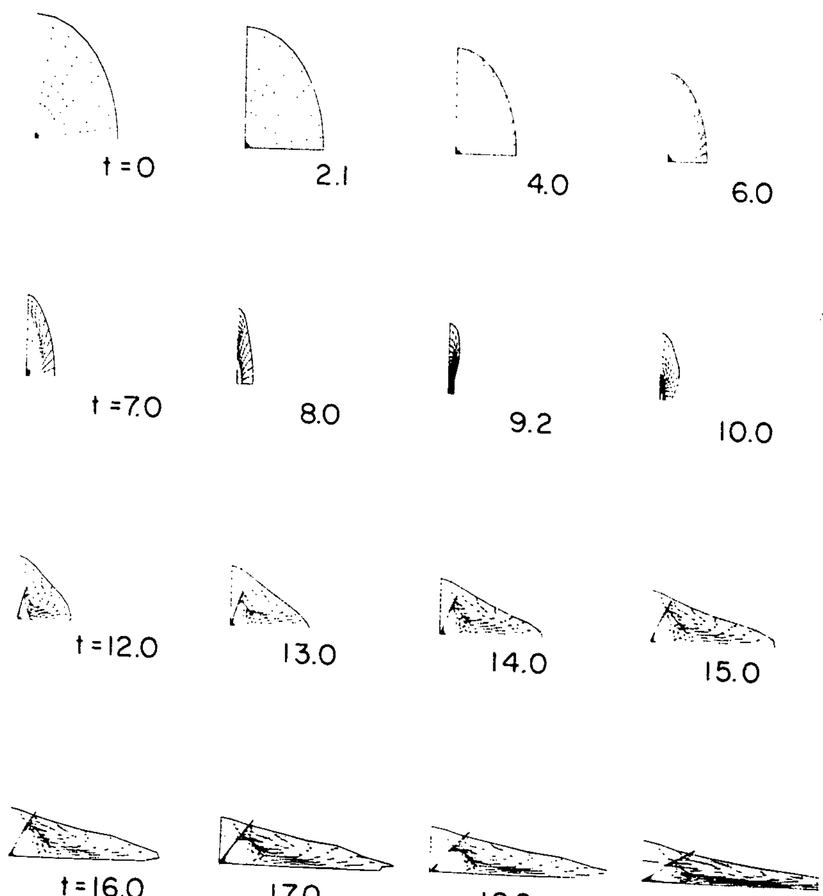

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Fig. 10. Cathode ray tube (CRT) plots at various times of Lagrangian grid laid out on the system of the sample problem.

END